MATH 2060B Tutorial 8

7.4 Q8 Let f be continuous on $I = [a,b]$ and assume $f(x) \geq 0$ $\forall x \in I$. Prove that if $L(f) = 0$, then $f(x) = 0$ for all $x \in I$.

Pf. Suppose for a contradiction, $f(c) > 0$ for some $c \in (a,b)$.
By definition of continuity of f at c, (with $\varepsilon = f(c)/2 > 0$)
there exists $\delta > 0$ such that $(c-\delta, c+\delta) \subset [a,b]$
and for all $x \in (c-\delta, c+\delta)$,

$$|f(x) - f(c)| < f(c)/2$$

$$\Rightarrow \quad f(x) > f(c)/2 > 0.$$

Consider the partition $\mathcal{P} = (a, c-\delta, c+\delta, b)$. Then

$$L(f;\mathcal{P}) = \underbrace{\inf_{x\in[a,c-\delta]} f(x)}_{\geq 0}(c-\delta-a) + \underbrace{\inf_{x\in[c-\delta,c+\delta]} f(x)}_{\geq f(c)/2}(2\delta) + \underbrace{\inf_{x\in[c+\delta,b]} f(x)}_{\geq 0}(b-c-\delta)$$

$$\geq \frac{f(c)}{2} \cdot 2\delta = f(c)\delta$$

Since $L(f) = \sup\{L(f,Q) : Q \text{ partition}\} \geq f(c)\delta > 0$,
this is a contradiction.

Thus $f(x) = 0$ for all $x \in (a,b)$. By continuity of f at a, b, we must have $f = 0$ on $[a,b]$. $\square$

7.4 Q15 Let f be defined on $I=[a,b]$ and assume that f satisfies the Lipschitz condition $|f(x)-f(y)|\le K|x-y|$ for $x,y\in I$. If P_n is the partition of I into n equal parts, show that $0\le U(f;P_n)-\int_a^b f \le \frac{K(b-a)^2}{n}$.

Pf. Let the n subintervals be $I_1,\ldots,I_n$. Because f continuous, there exists $u_j, v_j\in I_j$ such that

$$f(u_j)=\inf_{x\in I_j} f(x) \quad , \quad f(v_j)=\sup_{x\in I_j} f(x).$$

Note that $|v_j-u_j|\le \frac{b-a}{n}$.

By definition,

$$U(f,P_n)=\sum_{j=1}^n \sup_{x\in I_j} f(x)\left(\frac{b-a}{n}\right) \qquad L(f,P_n)=\sum_{j=1}^n \inf_{x\in I_j} f(x)\left(\frac{b-a}{n}\right)$$

$$=\frac{b-a}{n}\sum_{j=1}^n f(v_j) \qquad\qquad =\frac{b-a}{n}\sum_{j=1}^n f(u_j)$$

So

$$U(f,P_n)-L(f,P_n)=\frac{b-a}{n}\sum_{j=1}^n |f(v_j)-f(u_j)|$$

$$\le \frac{b-a}{n}\sum_{j=1}^n K|v_j-u_j|$$

$$\le \frac{b-a}{n}\sum_{j=1}^n \frac{K(b-a)}{n}$$

$$=\frac{K(b-a)^2}{n}.$$

Because $L(f;P_n)\le \int_a^b f \le U(f;P_n)$, we have

$$0\le U(f;P_n)-\int_a^b f \le U(f,P_n)-L(f;P_n)\le \frac{K(b-a)^2}{n}.$$

$\square$

8.1 Q23 Let $(f_n), (g_n)$ be sequences of bounded functions on A that converge uniformly on A to f, g respectively. Show that $(f_n g_n)$ converges uniformly on A to fg.

Claim. Suppose (h_n) is a sequence of bounded functions converging uniformly to h on A. Then

(1) (h_n) is uniformly bounded (i.e. $\exists M$ s.t. $|h_n(x)| \le M$ $\forall x, n$)

(2) h is also bounded.

Pf of Claim

For (1), Suppose $|h_n| \le M_n$ for each n.

Take $\varepsilon = 1$ in Cauchy criterion for uniform convergence

$\Rightarrow \exists N \in \mathbb{N}$ such that $\forall m \ge N$

$$|h_m(x) - h_N(x)| \le 1 \qquad \forall x \in A.$$

$\Rightarrow \forall m \ge N, \forall x \in A,$

$$|h_m(x)| \le |h_m(x) - h_N(x)| + |h_N(x)|$$
$$\le 1 + M_N$$

Therefore, for all $n \in \mathbb{N}$, $x \in A$

$$|h_n(x)| \le \max\{M_1, \ldots, M_{N-1}, M_N + 1\} =: M.$$

This proves (1). (2) follows by taking $n \to \infty$. ■

Pf of Q23 By above, $\exists M$ such that $|f_n|, |g_n|, |f|, |g| < M$.

Fix $\varepsilon > 0$, take $N \in \mathbb{N}$ such that for all $n \ge N$

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2M}, \quad |g_n(x) - g(x)| < \frac{\varepsilon}{2M} \qquad \forall x \in A.$$

For $x \in A$, $n \ge N$

$$\begin{aligned}
|f_n(x)g_n(x) - f(x)g(x)| &\le |(f_n(x) - f(x))g_n(x) + f(x)(g_n(x) - g(x))| \\
&\le |f_n(x) - f(x)||g_n(x)| + |f(x)||g_n(x) - g(x)| \\
&\le \frac{\varepsilon}{2M} \cdot M + M \cdot \frac{\varepsilon}{2M} \\
&= \varepsilon.
\end{aligned}$$

Therefore $(f_n g_n)$ converges to fg uniformly. $\square$