MATH2060 Solution 8

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8.1 Q22

(f_n) converges to f uniformly because $\left|f_n(x) - f(x)\right| = \frac{1}{n}$ for all $x \in \mathbb{R}$ so that $\|f_n - f\|_{\mathbb{R}} = \frac{1}{n} \to 0$ as $n \to \infty$. It is clear that the pointwise limit of (f_n^2) is f^2. But

$$\left|f_n^2(x) - f^2(x)\right| = \left|\frac{2x}{n} + \frac{1}{n^2}\right|$$

is unbounded for each n, so the convergence $f_n^2 \to f^2$ cannot be uniform. (For concreteness, one may take $x_n = n$ and observe that $\left|f_n^2(x_n) - f^2(x_n)\right| = 2 + 1/n^2 > 2$ for all n, so the convergence is not uniform by definition.)

8.2 Q4

Fix $\epsilon > 0$. By continuity of f, take $\delta > 0$ so that for all $x \in I$ with $|x - x_0| < \delta$, we have $\left|f(x) - f(x_0)\right| < \epsilon/2$. Since $x_n \to x_0$, choose N_1 such that $|x_n - x_0| < \delta$ for all $n \geq N_1$; by uniform convergence of (f_n) to f, choose N_2 such that $\left|f_n(x) - f(x)\right| < \epsilon/2$ for all $x \in I$ and $n \geq N_2$. Then for any $n \geq N := \max\{N_1, N_2\}$

$$\left|f_n(x_n) - f(x_0)\right| \leq \left|f_n(x_n) - f(x_n)\right| + \left|f(x_n) - f(x_0)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $\lim_{n\to\infty} f_n(x_n) = f(x_0)$.

8.2 Q7

By uniform convergence of (f_n) to f (with $\epsilon = 1$), there exists $N \in \mathbb{N}$ such that $\left|f(x) - f_N(x)\right| < 1$ for all $x \in A$. So for any $x \in A$, $\left|f(x)\right| \leq \left|f(x) - f_N(x)\right| + \left|f_N(x)\right| < 1 + M_N$ (where M_N is a bound of f_N on A), i.e. f is bounded on A.

8.2 Q8

For each n, $f_n(x)$ is bounded by n on $[0, \infty)$. Indeed, if $0 \leq x \leq 1$, $0 \leq f_n(x) \leq nx \leq n$; while if $x > 1$, $0 \leq f_n(x) \leq \frac{1}{x} < 1$. At $x = 0$, $f_n(0) = 0$ for all n, and thus $f_n(0) \to 0$; while for $x > 0$, $f_n(x) = \frac{1}{1/(nx)+x} \to 1/x$ as $n \to \infty$, so (f_n) converges pointwise to the function

$$f(x) = \begin{cases} 0 & \text{for } x = 0 \\ \frac{1}{x} & \text{for } x > 0 \end{cases},$$

which is unbounded on $[0, \infty)$. (f_n) does not converge to f uniformly by Question 7, or by observing that f is not continuous.

8.2 Q15

The argument of Example 8.1.9 (e) shows that g_n attains a maximum on $[0, 1]$ at $x_n = \frac{1}{n+1}$. This implies that

$$\|g_n\|_{[0,1]} = g_n(x_n) = \frac{n}{n+1} \cdot \left(1 + \frac{1}{n}\right)^{-n},$$

which converges to $1/e \neq 0$ as $n \to \infty$. So (g_n) does not converge uniformly. But observe that (g_n) converges pointwise to $g = 0$. Indeed, $g_n(0) = g_n(1) = 0$ for all n; and for each $0 < x < 1$, Theorem 3.2.11 ('ratio test' for sequences) implies that $\lim_{n\to\infty} g_n(x) = 0$ because

$$\lim_{n\to\infty} \frac{g_{n+1}(x)}{g_n(x)} = \lim_{n\to\infty} \frac{(n+1)x(1-x)^{n+1}}{nx(1-x)^n} = 1 - x < 1.$$

Since g_n and g are all integrable on $[0, 1]$, and $\|g_n\|_{[0,1]} \leq 1$, the bounded convergence theorem implies that $\lim_n \int_0^1 g_n = \int_0^1 \lim_n g_n = 0$.

8.2 Q17

Clearly each f_n is discontinuous at 0, and $f_n \leq f_m$ for $n \geq m$, since then $(0, 1/n) \subseteq (0, 1/m)$. The pointwise limit of (f_n) is $f = 0$, because for each fixed x, the sequence $f_n(x)$ is eventually identical to 0. f is continuous but the convergence is not uniform on $[0, 1]$ because $\|f_n - f\|_{[0,1]} = \|f_n\|_{[0,1]} = 1$ for all n.