1 More Topological Notions on $\mathbb{R}$

1.1 Compactness

Definition 1.1. Let $A \subset \mathbb{R}$ be a subset. The the following are equivalent

i. A is compact

ii. Every sequence in A has a convergent subsequence converging in A. (sequential compactness)

iii. A is closed and bounded. (Heine Borel Characterization)

iv. If $\{I_i\}_{i \in I}$ is an open (interval) cover of A, that is, $A \subset \bigcup_{i \in I} I_i$ where I is some index set, then there exists a **finite** subset $F \subset I$ such that $A \subset \bigcup_{i \in F} I_i$. (Open Cover Characterization)

Theorem 1.2 (Continuity Preserves Compactness)**.** *Let $f : A \to \mathbb{R}$ be a continuous function where A is compact. Then $f(A)$ is a compact subset.*

Corollary 1.3 (Extreme Value Theorem)**.** *Let $f : A \to \mathbb{R}$ be a continuous function from some compact set A. Then $f(A)$ is bounded and we have*

$$\sup_{x \in A} f(A) = \max_{x \in A} f(A) \qquad \inf_{x \in A} f(A) = \min_{x \in A} f(A)$$

Example 1.4. *Show that there is no continuous map from $[0, 1]$ onto $(0, 1)$.*

Solution. Suppose not. Let $f : [0, 1] \to (0, 1)$ be such continuous function. Then $f([0, 1]) = (0, 1)$ is a compact set. However $(0, 1)$ is not compact as it is not closed.

Example 1.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that $\lim_{x\to\infty} f(x), \lim_{x\to-\infty} f(x) < \infty$. Show that f is bounded.*

Solution. Write $L := \lim_{x\to-\infty} f(x)$ and $R := \lim_{x\to\infty} f(x)$. Then by definition, there exists $A, B > 0$ such that $|f(x)| \le \max\{1 + |L|, 1 + |R|\}$ on $(-\infty, -A)$ and (B, ∞). It remains to show that f is bounded on $[A, B]$. This follows from the extreme value theorem immediately.

1.2 Connectedness

Definition 1.6. Let $A \subset \mathbb{R}$. Then the following are equivalent

i. A is an interval, that is, in the form of $[a, b], (a, b], [a, b), (a, b)$ where $a, b \in \mathbb{R}$ and $a, b \in \{\pm\infty\}$ for the open case.

ii. For all $x, y \in A$, we have $[x, y] \subset A$.

Remark. Singletons are intervals in the form $[x, x]$ from this definition. We call them non-empty degenerate intervals.

Theorem 1.7 (Intermediate Value Theorem)**.** *Let I be an interval and $f : I \to \mathbb{R}$. Let $x, y \in I$ and $c \in \mathbb{R}$ such that $f(x) < c < f(y)$. Then there exists $z \in (x, y)$ or (y, x) such that $f(z) = c$. Equivalently, Let I be an interval and $f : I \to \mathbb{R}$ be continuous. Then $f(I)$ is an interval.*

Example 1.8. *Let $f : I \to \mathbb{R}$ be a continuous function from some interval. Suppose $f(I) \subset \mathbb{Q}$. Is it a must that f is a constant?*

Solution. Yes, it is. Note that $f(I)$ is an interval. Suppose $f(I)$ is non-degenerate. Then I contains some non-empty open interval (why?). It follows that $f(I) \cap \mathbb{R} \backslash \mathbb{Q} \neq \phi$ by denseness of irrational numbers. Hence $f(I)\mathbb{Q}$. It must be the case that $f(I)$ is degenerate, that is, a singleton.

Quick Practice.

1. *Let $A \subset \mathbb{R}$ be a compact set.*
   (a) *Let (x_n) be a sequence in A. Show that (x_n) converges if each of its convergent subsequence converges to the same limit.*
   (b) *Let $f : A \to f(A)$ be a continuous bijection. Show that the inverse f^{-1} is continuous.*
2. *Prove or disprove the following:*
   (a) *There exists a continuous function from $[0, 1]$ onto $(0, 1)$*
   (b) *There exists a continuous function from $(0, 1)$ onto $[0, 1]$*
   (c) *There exists a continuous* ***bijection*** *from $(0, 1)$ onto $[0, 1]$.* *(Hint: Only 1 of them is correct)*

2 Monotone Functions and Homeomorphisms of Intervals

Definition 2.1. Let $f : A \to \mathbb{R}$ be a function. Then

- f is increasing if $f(x) \le f(y)$ for all $x, y \in A$ with $x \le y$.
- f is strictly increasing if $f(x) < f(y)$ for all $x, y \in A$ with $x < y$
- Similarly one can define (strictly) decreasing functions and functions are called (strictly) monotone if it is either (strictly) increasing or (strictly) decreasing

Theorem 2.2. *Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be an increasing function. Then for all $x \in (a, b)$ $f(x^+) := \lim_{t \to x^+} f(t)$ and $f(x^-) := \lim_{t \to x^-} f(t)$ exist. Using similar notations, we also have $f(a^+), f(b^-) < \infty$*

Proof. Note that f is bounded. Then we leave it as an exercise for readers to show that for all $x \in (a, b)$, we have $f(x^+) = \inf\{f(t) : t > x\}$ and $f(x^-) = \sup\{f(t) : t < x\}$. The case for endpoints is similar. □

Definition 2.3. Let $f : A \to f(A) \subset \mathbb{R}$ be a function and A a subset of $\mathbb{R}$. Then we call f to be a homeomorphism if f is a continuous bijection onto its image such that f^{-1} is continuous.

The proof of the following is left as an exercise.

Theorem 2.4. *Let $f : [0, 1] \to \mathbb{R}$ be a homeomorphism onto its image such that $f(0) < f(1)$. Then f is strictly increasing.*

3 Exercise

1. Let $f : [0, 1] \to \mathbb{R}$ be a homeomorphism onto its image.
   (a) Show that $f([0, 1]) \subset [f(0), f(1)]$ or $f([0, 1]) \subset [f(1), f(0)]$.
   (b) Show that f is strictly monotone.
   (c) Show that f is strictly monotone if the domain is replaced by any interval.
2. Let $K \subset \mathbb{R}$ be a subset. Show that K is a compact set if and only if every continuous function $f : K \to \mathbb{R}$ defined on K is bounded. *(Hint: Find examples of continuous unbounded functions if K is not compact)*
3. Let $A \subset \mathbb{R}$. Let $U \subset A$. We say that U is an open set with respect to A if $U = \mathcal{O} \cap A$ where $\mathcal{O}$ is an open set of $\mathbb{R}$. A subset $A \subset \mathbb{R}$ is said to be *connected* if A cannot be written as disjoint union of two proper subsets that are open with respect to A.
   (a) Show that $\mathbb{R}$ is connected. In other words, if $\mathbb{R} = U_1 \sqcup U_2$ where U_1, U_2 are open sets. Then $U_i = \mathbb{R}$ for some i.
   (b) Show that if $A \subset \mathbb{R}$ is connected, then it is an interval. *(Hint: Consider the contrapositive.)*
   (c) Show that every interval is connected. Therefore, connected subsets of $\mathbb{R}$ are precisely intervals.
4. Let $f : [0, 1] \to \mathbb{R}$ be an increasing function.
   (a) Show that f is discontinuous at $c \in (0, 1)$ if and only if $f(c^+) \neq f(c^-)$ where $f(c^+) := \lim_{x \to c^+} f(x)$ and $f(c^-) := \lim_{x \to c^-} f(x)$
   (b) Show that f has at most countably many point of discontinuity.
   (Hint: Consider the sets $F_\epsilon := \{c \in (0, 1) : f(c^+) - f(c^-) > \epsilon\}$ where $\epsilon > 0$)
5. Consider a function $f : \mathbb{R} \to \mathbb{R}$. Let $[a, b] \subset \mathbb{R}$ be an interval. A partition of $[a, b]$ is a finite sequence $(t_0, \cdots, t_n) \subset [a, b]$ where $t_0 := a$ and $t_n := b$. We define the variation of f over $[a, b]$ to be
   $$V_f([a, b]) := \sup\{\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| : \{t_i\} \subset [a, b] \text{ a partition}\}$$
   We say that f has finite variation over $[a, b]$ if $V_f([a, b]) < \infty$
   (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone function. Show that f has finite variation over $[0, t]$ for all $t > 0$.
   (b) Let $f : [0, \infty) \to \mathbb{R}$ be a function such that $V_f(t) := V([0, t])$ is finite for all $t \ge 0$ (which is also called a function of finite variation). Show that $V_f(t) : [0, \infty) \to \mathbb{R}$ is an increasing function.
   (c) Show that if f is of finite variation (which satisfies the conditions in (b)). Then for all $x \in (0, \infty)$, $f(x^+) := \lim_{t \to x} f(t)$ and $f(x^-) := \lim_{t \to x} f(t)$ exist and f has at most countably many discountinuity.