1 Uniform Continuity

Definition 1.1. Let $f : A \to \mathbb{R}$ be a function where $A \subset \mathbb{R}$. Then we call f to be *uniformly continuous* if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ would imply $|f(x) - f(y)| < \epsilon$.

Remark. Every uniformly continuous function is continuous on its domain. This can be checked from definition.

Example 1.2. *Define $f(x) := \sqrt{x}$ for all $x \geq 0$. Show that*

a. f is uniformly continuous on $[a, 1]$ for all $a > 0$.

b. f is uniformly continuous on $[0, 1]$

Solution. a. Let $\epsilon > 0$. We first observe that for all $x, y \in [a, 1]$, we have $|f(x) - f(y)| = \left|\sqrt{x} - \sqrt{y}\right| = |x - y| \frac{1}{\sqrt{x}+\sqrt{y}} \leq |x - y| \frac{1}{\sqrt{a}+\sqrt{a}} = |x - y| \frac{1}{2\sqrt{a}}$. Hence by taking $\frac{\delta}{2\sqrt{a}} < \epsilon$, we have $|f(x) - f(y)| \leq \epsilon$ for all $x, y \in [a, 1]$.

b. Let $\epsilon > 0$. First note that $\lim_{x \to 0^+} f(x) = 0$. Hence there exists $t > 0$ such that $|f(x)| < \epsilon/2$. It follows that for all $x, y \in [0, t)$, we have $|f(x) - f(y)| < \epsilon$. Next, f is continuous at $t \geq 0$. Hence, there exists $\delta_1 > 0$ such that if $|x - t| < \delta_1$ and $x \geq 0$, we have $|f(x) - f(t)| < \epsilon/2$. Furthermore, f is uniformly continuous on $[t, 1]$ by the first part. Therefore, there exists $\delta_2 > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in [t, 1]$ with $|x - y| < \delta_2$. Now take $\delta < \delta_1, \delta_2, t$. Then for all $x, y \in [0, 1]$ with $|x - y| < \delta$, either $x, y \in [0, t)$, $x, y \in B(t, \delta_1)$ or $x, y \in [t, 1]$. In any case, we have $|f(x) - f(y)| < \epsilon$.

Theorem 1.3 (Uniformly Continuous Theorem)**.** *Let $f : A \to \mathbb{R}$ be a continuous function with $A \subset \mathbb{R}$. If A is a compact set, then f is uniformly continuous.*

Example 1.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a funciton. Suppose $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist. Show that f is uniformly continuous.*

Solution. Write $L := \lim_{x\to-\infty} f(x)$ and $R := \lim_{x\to\infty f(x)}$. Let $\epsilon > 0$. Then there exists $a, b \in \mathbb{R}$ such that $|f(x) - L| < \epsilon/2$ for all $x \in (-\infty, a)$ and $|f(x) - R| < \epsilon/2$ for all $x \in (b, \infty)$. It follows from the triangle inequality that $|f(x) - f(y)| < \epsilon$ when either $x, y \in (-\infty, a)$ or $x, y \in (b, \infty)$. Now note that $[a, b]$ is a compact interval. Therefore, f is uniformly continuous of $[a, b]$ by the Uniformly Continuous Theorem. Therefore, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ when $|x - y| < \delta$ with $x, y \in [a, b]$. Note that f is continuous at a, b. Hence, there exist $\delta_a, \delta_b > 0$ such that $|f(x) - f(a)| < \epsilon/2$ for $x \in B(a, \delta_a)$ and $|f(x) - f(b)| < \epsilon/2$ for $x \in B(b, \delta_b)$. Now take $\delta < \delta_a, \delta_b$. Let $x, y \in \mathbb{R}$ with $|x - y| < \delta$. Then it must be the case that $x, y \in (-\infty, a)$, $x, y \in (b, \infty)$, $x, y \in [a, b]$ or x, y lying in the δ_a or δ_b neighborhood of a, b. It follows that $|f(x) - f(y)| < \epsilon$

Proposition 1.5 (Divergence Criteria for Uniform Continuity)**.** *Let $f : A \to \mathbb{R}$ be a function. Then f is not uniformly continuous if and only if there exist sequences (x_n) and (y_n) in A with $\lim |x_n - y_n| = 0$, and $\epsilon_0 > 0$ such that $|f(x_n) - f(y_n)| \geq \epsilon_0$*

Example 1.6. *Define $f(x) := 1/x$ for all $x \geq 0$. Show that f is not uniformly continuous on $(0, 2)$.*

Solution. Consider $(x_n := 1/n)$ and $(y_n := 1/n^2)$. Then $\lim |x_n - y_n| = \left|1/n - 1/n^2\right| = 0$. However, we have $|f(x_n) - f(y_n)| = \left|n^2 - n\right| = n(n - 1) \geq n \geq 1$ for all $n \geq 2$. It follows by considering suitable tail subsequences that the divergence criteria is satisfied. Hence f is not uniformly continuous on $(0, 2)$

Quick Practice.

1. *For each of the following, f is a real-valued function defined on a subset $A \subset \mathbb{R}$. Determine if f is uniformly continuous on A by definition.*

   a) $f(x) = x^2$, $A = [0, 1]$
   b) $f(x) = x^2$, $A = \mathbb{R}$
   c) $f(x) = \frac{1}{x-3}$ $A = \mathbb{R}\backslash\{3\}$
   d) $f(x) = \sin(1/x)$, $A = (0, \infty)$
   e) $f(x) = x \sin x$, $A = \mathbb{R}$
   f) $f(x) = \inf\{|y - x| : y \notin \mathbb{Q}\}$, $A = \mathbb{R}$

2. *(P. 164, Q9). Let $f : A \to \mathbb{R}$ be uniformly continuous such that $\inf\{|f(x)| : x \in A\} > 0$. Show that $1/f$ is uniformly continuous on A.*

3. *(P. 164, Q10). Let $f : A \to \mathbb{R}$ be uniformly continuous. Suppose A is bounded then $f(A)$ is bounded.*

4. *(P. 164, Q12). Let $f : [0, \infty) \to \mathbb{R}$ be continuous. Suppose f is uniformly continuous on $[a, \infty)$ for some $a > 0$. Show that f is uniformly continuous on $[0, \infty)$*

5. *(Uniform Continuous Extension Theorem) Let $f : \mathbb{Q} \to \mathbb{R}$ be a uniformly continuous function.*

   (a) Show that if $g, h : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $g\mid_{\mathbb{Q}} = h\mid_{\mathbb{Q}} = f$. Then $g = h$ on $\mathbb{R}$.
   *(b) Show that there **exists** a unique continuous function $\overline{f} : \mathbb{R} \to \mathbb{R}$ such that $\overline{f} = f$ on $\mathbb{Q}$.*
   (c) Show that $\overline{f}$ is uniformly continuous on $\mathbb{R}$.
   (d) Is (b) true for continuous f in general?

2 Lipschitz Functions

Definition 2.1. Let $f : A \to \mathbb{R}$ be a function. Then we say f to be *Lipschitz* on A if there exists $L > 0$ such that we have for all $x, y \in A$ that

$$|f(x) - f(y)| \leq L|x - y|$$

Remark. A Lipschitz function is uniformly continuous and hence continuous.

Example 2.2. *Show that* $f(x) := x^2$ *uniformly continuous on* $[0, 1]$.

Solution. Let $x, y \in [0, 1]$. We have $|f(x) - f(y)| = \left|x^2 - y^2\right| = |x - y||x + y| \leq 2|x - y|$. Hence, f is Lipschitz on $[0, 1]$. It follows that f is uniformly continuous.

Example 2.3. *Show that* $f(x) := \sqrt{x}$ *is uniformly continuous, but not Lipschitz on* $[0, 1]$

Solution. The uniform continuity has been shown on last page. Now suppose f were Lipschitz. Then there exists $C > 0$ such that for all $x \in (0, 1]$, we have $|f(x) - f(0)| = \sqrt{x} \leq C|x|$. Hence, it follows that $C^{-1} \leq \sqrt{x}$ for all $x \in [0, 1]$. This implies that $\inf\{\sqrt{x} : x \in (0, 1]\} > 0$, which is a contradiction (why?).

3 Exercise

1. Let $f : A \to \mathbb{R}$ be a function. We say that f satisfies property (P) if there exists an increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\lim_{t\to 0^+} \phi(t) = 0$ such that for all $x, y \in A$

$$|f(x) - f(y)| \leq \phi(|x - y|)$$

   Show that every function that satisfies property (P) (on its domain) is uniformly continuous.

2. Let $f : A \to \mathbb{R}$ be a function. Define

$$\omega_f(t) := \sup\{|f(x) - f(y)| : |x - y| \leq t, x, y \in A\} \in [0, \infty]$$

   (a) Show that if f is uniformly continuous, then $\omega_f(t) < \infty$ for all $t \geq 0$

   (b) Show that a function is uniformly continuous if and only if property (P) (defined in Q1) is satisfied.

   (c) Show that f is Lipschitz on A if and only if there exists $L > 0$ such that $\omega_f(t) \leq Lt$ for all $t \in [0, \infty)$. Hence, show that every Lipschitz function is uniformly continuous.

3. Let $f : A \to \mathbb{R}$. Then we define $\text{Lip}(f) := \sup\{\frac{|f(x)-f(y)|}{|x-y|} : x \neq y \in A\} \in [0, \infty]$.

   (a) Show that $\text{Lip}(f) < \infty$ if and only if f is Lipscthiz. Furthermore, if this is the case, we have $\text{Lip}(f) := \inf\{L > 0 : |f(x) - f(y)| \leq L|x - y|\}$

   (b) Show that $\text{Lip}(f) = 0$ if and only if f is a constant function.

   (c) Let $f, g : A \to \mathbb{R}$ be Lipschitz. Show that $f + g, \max\{f, g\}$ and $\min\{f, g\}$ are Lipschitz functions. Furthermore, $\text{Lip}(f + g) \leq \text{Lip}(f) + \text{Lip}(g)$ and $\text{Lip}(\max\{f, g\}) \leq \max\{\text{Lip}(f), \text{Lip}(g)\}\}$

   (d) Show that $\text{Lip}(fg) \leq \text{Lip}(f) \sup\{|g(x)| : x \in A\} + \text{Lip}(g) \sup\{|f(x)| : x \in A\}$ where we allows the supremums to be ∞ for every f, g that is Lipschitz.

   (e) Given an example that f, g are Lipschitz but the point-wise product fg is not.

   Remark. For (b), we have $\max\{f, g\}(x) := \max\{f(x), g(x)\}$ for all $x \in A$. The minimum is defined similarly.

4. We say that a function $f : A \to \mathbb{R}$ is a bi-Lipschitz function if there exist $C_1, C_2 > 0$ such that for all $x, y \in A$

$$C_1|x - y| \leq |f(x) - f(y)| \leq C_2|x - y|$$

   (a) Let $f : A \to \mathbb{R}$ be a bi - Lipschitz function. Show that f is injective. Furthermore, $f : A \to f(A)$ and $f^{-1} : f(A) \to A$ are Lipschitz functions.

   (b) Show that if $f : A \to \mathbb{R}$ is a bi-Lipschitz function, then A is bounded if and only if $f(A)$ is bounded. Furthermore A is closed if and only if $f(A)$ is closed.

   (c) Give examples to show that part (b) is not true if we relax f to be a homeomorphism onto its image, that is f is continuous with continuous inverse, instead of being bi-Lipschitz.
   (You may assume the continuity properties of functions that you come across in high schools)

5. (Uniformly continuous maps can be "Lipschitz-ized"). Let $f : A \to \mathbb{R}$ be uniformly continuous. Show that for all $\theta > 0$, there exists $K(\theta)$ such that if $x, y \in A$ are with $|x - y| \geq \theta$, then $|f(x) - f(y)| \leq K(\theta)|x - y|$.

6. (Lipschitz Extension Theorem/ Mc-Shane Extension Theorem) Let $f : A \to \mathbb{R}$ be a Lipschitz function. For all $a \in A$, define $g(x) := f(a) + \text{Lip}(f)|x - a|$ for all $x \in \mathbb{R}$. Define $F : \mathbb{R} \to \mathbb{R}$ by considering the infimum $F(x) := \inf\{g_a(x) : a \in A\}$ for all $x \in \mathbb{R}$. Show that F is a Lipschitz function extending f ($F|_A = f$) such that $\text{Lip}(F) = \text{Lip}(f)$.

7. This exercise gives a (short) proof of the uniform continuous theorem on a compact interval. Let $f : I := [a, b] \to \mathbb{R}$ be continuous. Let $\epsilon > 0$. Define $S := \{c \in [a, b] : c > a, |f(x) - f(y)| < \epsilon \textit{ on } [a,\ c)\}$.

   (a) Show that $\sup S = b$.

   (b) Using (a), show that f is uniformly continuous on $[a, b]$.

8. This exercise gives a proof of the uniform continuous theorem on a compact interval using the monotone convergence theorem together with an *Exhaustion Argument*. Let $f : I := [a, b] \to \mathbb{R}$ be a continuous function. Let $\epsilon > 0$.

   (a) Show that there exists $c > a$ such that for all $x, y < c$ with $x, y \in I$ we have $|f(x) - f(y)| < \epsilon$.

   (b) Define $A_1 := \{j \in \mathbb{N} \mid \exists c : c - a > \frac{1}{j}, |f(x) - f(y)| < \epsilon \textit{ on } [a, c)\} \subset \mathbb{N}$. Show that $j_1 := \min A_1$ exists. (*Hint: You may use the well-order property of* $\mathbb{N}$)

   (c) Define $a_1 > a$ such that $a_1 - a > 1/j_1$ and $|f(x) - f(y)| < \epsilon$ on $[a, a_1)$.
   Define $A_2 := \{j \in \mathbb{N} : c - a_1 > \frac{1}{j}, |f(x) - f(y)| < \epsilon \textit{ on } [a, c)\} \subset \mathbb{N}$. Show that $j_2 := \min A_2$ exists.

   (d) Show that there exists a strictly increasing sequence (a_n) in I a sequence of natural numbers (j_n) such that if $a_0 := a$, we have

      i. $\dfrac{1}{j_{n+1} + 1} \geq a_{n+1} - a_n > \dfrac{1}{j_{n+1}}$ for all $n \geq 0$

      ii. if $c > a$ is such that $c - a_n > 0$ and $|f(x) - f(y)| < \epsilon$ on $[a, c)$, then we have $\dfrac{1}{j_{n+1} + 1} \geq c - a_n$

   (e) Define $L := \lim a_n$ by the bounded monotone convergence theorem, show that $L = b$.

   (f) Hence, show that f is uniformly continuous on $[a, b]$.

   Remark. The technique used in this question is called **Exhaustion Argument** because the sequence (a_n) is defined by considering the most optimal objects we can construct.

9. Let $f : A \to \mathbb{R}$ be a function on some subset. We say that f is lower semi-continuous (lsc.) at $x \in A$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|y - x| > \delta$ with $y \in A$, then $f(x) - f(y) < \epsilon$.

   a. Show that f is lower semi-continuous at $x \in A$ if and only if sequences (x_n) in A with $\lim x_n = x$, we have $f(x) \leq \liminf f(x_n)$.

   b. We say that f is lower semi-continuous on A if it is at every point of A. Suppose A is compact and f is lower semi-continuous. Show that f bounded below and minimum is attained, that is,

$$\inf\{f(x) : x \in A\} = \min\{f(x) : x \in A\}$$

10. Let $f : A \to \mathbb{R}$ be a lower semi-continuous function (see the previous question for the definition).

    a. For all $n \in \mathbb{N}$, define $f_n(x) := \inf\{f(y) + n|x - y| : y \in X\}$ for all $x \in A$. Show that f_n are well-defined continuous functions on A

    b. Show that there exists an increasing sequence of continuous functions $(g_n : A \to \mathbb{R})$, that is, $g_n(x) \leq g_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in A$, such that $\lim_n g_n(x) = f(x)$ for all $x \in A$