1 Uniform Continuity

Definition 1.1. Let \( f : A \to \mathbb{R} \) be a function where \( A \subset \mathbb{R} \). Then we call \( f \) to be uniformly continuous if and only if for all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( |x - y| < \delta \) would imply \( |f(x) - f(y)| < \epsilon \).

Remark. Every uniformly continuous function is continuous on its domain. This can be checked from definition.

Example 1.2. Define \( f(x) := \sqrt{x} \) for all \( x \geq 0 \). Show that

a. \( f \) is uniformly continuous on \([a, 1]\) for all \( a > 0 \).

b. \( f \) is uniformly continuous on \([0, 1]\)

Solution. a. Let \( \epsilon > 0 \). We first observe that for all \( x, y \in [a, 1] \), we have \( |f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq |x - y|\frac{1}{2\sqrt{a}} \). Hence by taking \( \delta = \frac{\epsilon}{2\sqrt{a}} < \epsilon \), we have \( |f(x) - f(y)| \leq \epsilon \) for all \( x, y \in [a, 1] \).

b. Let \( \epsilon > 0 \). First note that \( \lim_{x \to a^+} f(x) = 0 \). Hence there exists \( t > 0 \) such that \( |f(x)| < \epsilon/2 \). It follows that for all \( x, y \in [0, t] \), we have \( |f(x) - f(y)| < \epsilon \). Next, \( f \) is continuous at \( t \geq 0 \). Hence, there exists \( \delta_1 > 0 \) such that \( |x - t| < \delta_1 \) and \( x, y \geq 0 \), we have \( |f(x) - f(t)| < \epsilon/2 \). Furthermore, \( f \) is uniformly continuous on \([t, 1]\) by the first part. Therefore, there exists \( \delta_2 > 0 \) such that \( |f(x) - f(y)| < \epsilon \) for all \( x, y \in [t, 1] \) with \( |x - y| < \delta_2 \). Now take \( \delta < \delta_1, \delta_2 \). Then for all \( x, y \in [0, 1] \) with \( |x - y| < \delta \), either \( x, y \in [0, t] \), \( x, y \in B(t, \delta_1) \) or \( x, y \in [t, 1] \). In any case, we have \( |f(x) - f(y)| < \epsilon \).

Theorem 1.3 (Uniformly Continuous Theorem). Let \( f : A \to \mathbb{R} \) be a continuous function with \( A \subset \mathbb{R} \). If \( A \) is a compact set, then \( f \) is uniformly continuous.

Example 1.4. Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. Suppose \( \lim_{x \to -\infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \) exist. Show that \( f \) is uniformly continuous.

Solution. Write \( L := \lim_{x \to -\infty} f(x) \) and \( R := \lim_{x \to -\infty} f(x) \). Let \( \epsilon > 0 \). Then there exists \( a, b \in \mathbb{R} \) such that \( |f(x) - L| < \epsilon/2 \) for all \( x \in (-\infty, a) \) and \( |f(x) - R| < \epsilon/2 \) for all \( x \in (b, \infty) \). It follows from the triangle inequality that \( |f(x) - f(y)| < \epsilon \) when either \( x, y \in (-\infty, a) \) or \( x, y \in (b, \infty) \). Now note that \([a, b]\) is a compact interval. Therefore, \( f \) is uniformly continuous of \([a, b]\) by the Uniformly Continuous Theorem. Therefore, there exists \( \delta > 0 \) such that \( |f(x) - f(y)| < \epsilon \) when \( |x - y| < \delta \) with \( x, y \in [a, b] \). Note that \( f \) is continuous at \( a, b \). Hence, there exist \( \delta_a, \delta_b > 0 \) such that \( |f(x) - f(a)| < \epsilon/2 \) for \( x \in B(a, \delta_a) \) and \( |f(x) - f(b)| < \epsilon/2 \) for \( x \in B(b, \delta_b) \). Now take \( \delta < \delta_a, \delta_b \). Let \( x, y \in \mathbb{R} \) with \( |x - y| < \delta \). Then it must be the case that \( x \in (-\infty, a) \), \( x, y \in (b, \infty) \), \( x, y \in [a, b] \) or \( x, y \in \delta_a \) or \( \delta_b \) neighborhood of \( a, b \). It follows that \( |f(x) - f(y)| < \epsilon \).

Proposition 1.5 (Divergence Criteria for Uniform Continuity). Let \( f : A \to \mathbb{R} \) be a function. Then \( f \) is not uniformly continuous if and only if there exist sequences \((x_n) \) and \((y_n) \) in \( A \) with \( \lim |x_n - y_n| = 0 \), and \( \epsilon_0 > 0 \) such that \( |f(x_n) - f(y_n)| \geq \epsilon_0 \)

Example 1.6. Define \( f(x) := 1/x \) for all \( x \geq 0 \). Show that \( f \) is not uniformly continuous on \((0, 2)\).

Solution. Consider \( (x_n := 1/n) \) and \( (y_n := 1/n^2) \). Then \( \lim |x_n - y_n| = |1/n - 1/n^2| = 0 \). However, we have \( |f(x_n) - f(y_n)| = \left| n^2 - n \right| = n(n-1) \geq n \geq 1 \) for all \( n \geq 2 \). It follows by considering suitable tail subsequences that the divergence criteria is satisfied. Hence \( f \) is not uniformly continuous on \((0, 2)\)

Quick Practice.

1. For each of the following, \( f \) is a real-valued function defined on a subset \( A \subset \mathbb{R} \). Determine if \( f \) is uniformly continuous on \( A \) by definition.

a) \( f(x) = x^2 \), \( A = [0, 1] \)

b) \( f(x) = x^2 \), \( A = \mathbb{R} \)

c) \( f(x) = \frac{1}{x-3} \), \( A = \mathbb{R} \backslash \{3\} \)

d) \( f(x) = \sin(1/x) \), \( A = (0, \infty) \)

e) \( f(x) = x \sin x \), \( A = \mathbb{R} \)

f) \( f(x) = \inf \{|y-x| : y \notin \mathbb{Q} \} \), \( A = \mathbb{R} \)

2. (P. 164, Q9). Let \( f : A \to \mathbb{R} \) be uniformly continuous such that \( \inf \{|f(x)| : x \in A\} > 0 \). Show that \( 1/f \) is uniformly continuous on \( A \).

3. (P. 164, Q10). Let \( f : A \to \mathbb{R} \) be uniformly continuous. Suppose \( A \) is bounded then \( f(A) \) is bounded.

4. (P. 164, Q12). Let \( f : [0, \infty) \to \mathbb{R} \) be continuous. Suppose \( f \) is uniformly continuous on \([a, \infty)\) for some \( a > 0 \). Show that \( f \) is uniformly continuous on \([0, \infty)\)

5. (Uniform Continuous Extension Theorem) Let \( f : Q \to \mathbb{R} \) be a uniformly continuous function.

a) Show that if \( g, h : \mathbb{R} \to \mathbb{R} \) are continuous functions such that \( g \mid_Q = h \mid_Q = f \). Then \( g = h \) on \( \mathbb{R} \).

b) Show that there exists a unique continuous function \( \overline{f} : \mathbb{R} \to \mathbb{R} \) such that \( \overline{f} = f \) on \( \mathbb{Q} \).

c) Show that \( \overline{f} \) is uniformly continuous on \( \mathbb{R} \).

d) Is (b) true for continuous \( f \) in general?
2 Lipschitz Functions

Definition 2.1. Let \( f : A \to \mathbb{R} \) be a function. Then we say \( f \) to be Lipschitz on \( A \) if there exists \( L > 0 \) such that we have for all \( x, y \in A \) that
\[
|f(x) - f(y)| \leq L|x - y|
\]

Remark. A Lipschitz function is uniformly continuous and hence continuous.

Example 2.2. Show that \( f(x) := x^2 \) uniformly continuous on \([0, 1]\).

Solution. Let \( x, y \in [0, 1] \). We have \(|f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y| \leq 2|x - y|\). Hence, \( f \) is Lipschitz on \([0, 1]\). It follows that \( f \) is uniformly continuous.

Example 2.3. Show that \( f(x) := \sqrt{x} \) is uniformly continuous, but not Lipschitz on \([0, 1]\).

Solution. The uniform continuity has been shown on last page. Now suppose \( f \) were Lipschitz. Then there exists \( C > 0 \) such that for all \( x \in (0, 1] \), we have \(|f(x) - f(0)| = \sqrt{x} \leq C|x|\). Hence, it follows that \( C^{-1} \leq \sqrt{x} \) for all \( x \in (0, 1] \). This implies that \( \inf\{\sqrt{x} : x \in (0, 1]\} > 0 \), which is a contradiction (why?).

3 Exercise

1. Let \( f : A \to \mathbb{R} \) be a function. We say that \( f \) satisfies property \( (P) \) if there exists an increasing function \( \phi : [0, \infty) \to [0, \infty) \) with \( \lim_{t \to 0^+} \phi(t) = 0 \) such that for all \( x, y \in A \)
\[
|f(x) - f(y)| \leq \phi(|x - y|)
\]
Show that every function that satisfies property \( (P) \) (on its domain) is uniformly continuous.

2. Let \( f : A \to \mathbb{R} \) be a function. Define
\[
\omega_f(t) := \sup\{|f(x) - f(y)| : |x - y| \leq t, x, y \in A\} \in [0, \infty]
\]
(a) Show that if \( f \) is uniformly continuous, then \( \omega_f(t) < \infty \) for all \( t \geq 0 \)
(b) Show that a function is uniformly continuous if and only if property \( (P) \) (defined in Q1) is satisfied.
(c) Show that \( f \) is Lipschitz on \( A \) if and only if there exists \( L > 0 \) such that \( \omega_f(t) \leq Lt \) for all \( t \in [0, \infty) \).
Hence, show that every Lipschitz function is uniformly continuous.

3. Let \( f : A \to \mathbb{R} \). Then we define \( \text{Lip}(f) := \sup\{|f(x) - f(y)| : x \neq y \in A\} \in [0, \infty] \).
(a) Show that \( \text{Lip}(f) < \infty \) if and only if \( f \) is Lipschitz. Furthermore, if this is the case, we have \( \text{Lip}(f) := \inf\{L > 0 : |f(x) - f(y)| \leq L|x - y|\} \)
(b) Show that \( \text{Lip}(f) = 0 \) if and only if \( f \) is a constant function.
(c) Let \( f, g : A \to \mathbb{R} \) be Lipschitz. Show that \( f + g, \max\{f, g\} \) and \( \min\{f, g\} \) are Lipschitz functions. Furthermore, \( \text{Lip}(f + g) \leq \text{Lip}(f) + \text{Lip}(g) \) and \( \text{Lip}(\max\{f, g\}) \leq \max\{\text{Lip}(f), \text{Lip}(g)\} \)
(d) Show that \( \text{Lip}(fg) \leq \text{Lip}(f) \sup\{|g(x)| : x \in A\} + \text{Lip}(g) \sup\{|f(x)| : x \in A\} \) where we allow the supremums to be \( \infty \) for every \( f, g \) that is Lipschitz.
(e) Given an example that \( f, g \) are Lipschitz but the point-wise product \( fg \) is not.
Remark. For (b), we have \( \max\{f, g\}(x) := \max\{f(x), g(x)\} \) for all \( x \in A \). The minimum is defined similarly.

4. We say that a function \( f : A \to \mathbb{R} \) is a bi-Lipschitz function if there exist \( C_1, C_2 > 0 \) such that for all \( x, y \in A \)
\[
C_1|x - y| \leq |f(x) - f(y)| \leq C_2|x - y|
\]
(a) Let \( f : A \to \mathbb{R} \) be a bi - Lipschitz function. Show that \( f \) is injective. Furthermore, \( f : A \to f(A) \) and \( f^{-1} : f(A) \to A \) are Lipschitz functions.
(b) Show that if \( f : A \to \mathbb{R} \) is a bi-Lipschitz function, then \( A \) is bounded if and only if \( f(A) \) is bounded.
Furthermore \( A \) is closed if and only if \( f(A) \) is closed.
(c) Give examples to show that part (b) is not true if we relax \( f \) to be a homeomorphism onto its image, that is \( f \) is continuous with continuous inverse, instead of being bi-Lipschitz.
(You may assume the continuity properties of functions that you come across in high schools)
5. (Uniformly continuous maps can be "Lipschitz-ized"). Let \( f : A \to \mathbb{R} \) be uniformly continuous. Show that for all \( \theta > 0 \), there exists \( K(\theta) \) such that if \( x, y \in A \) are with \( |x - y| \geq \theta \), then \( |f(x) - f(y)| \leq K(\theta)|x - y| \).

6. (Lipschitz Extension Theorem/ Mc-Shane Extension Theorem) Let \( f : A \to \mathbb{R} \) be a Lipschitz function. For all \( a \in A \), define \( g(x) := f(a) + \text{Lip}(f)|x - a| \) for all \( x \in \mathbb{R} \). Define \( F : \mathbb{R} \to \mathbb{R} \) by considering the infimum \( F(x) := \inf\{g_a(x) : a \in A\} \) for all \( x \in \mathbb{R} \). Show that \( F \) is a Lipschitz function extending \( f \) (\( F |_A = f \)) such that \( \text{Lip}(F) = \text{Lip}(f) \).

7. This exercise gives a (short) proof of the uniform continuous theorem on a compact interval. Let \( f : I := [a, b] \to \mathbb{R} \) be continuous. Let \( \epsilon > 0 \). Define \( S := \{c \in [a, b] : c > a, |f(x) - f(y)| < \epsilon \text{ on } [a, c]\} \).
   (a) Show that \( \sup S = b \).
   (b) Using (a), show that \( f \) is uniformly continuous on \([a, b]\).

8. This exercise gives a proof of the uniform continuous theorem on a compact interval using the monotone convergence theorem together with an Exhaustion Argument. Let \( f : I := [a, b] \to \mathbb{R} \) be a continuous function. Let \( \epsilon > 0 \).
   (a) Show that there exists \( c > a \) such that for all \( x, y \in I \) we have \( |f(x) - f(y)| < \epsilon \).
   (b) Define \( A_1 := \{j \in \mathbb{N} : \exists \epsilon > 0 \text{ such that } \epsilon < |f(x) - f(y)| < \epsilon \text{ on } [a, c]\} \subset \mathbb{N} \). Show that \( j_1 := \min A_1 \) exists. (Hint: You may use the well-order property of \( \mathbb{N} \))
   (c) Define \( a_1 > a \) such that \( a_1 - a > 1/j_1 \) and \( |f(x) - f(y)| < \epsilon \text{ on } [a, a_1] \).
   Define \( A_2 := \{j \in \mathbb{N} : c - a_1 > \frac{1}{j_1}, |f(x) - f(y)| < \epsilon \text{ on } [a, a_1]\} \subset \mathbb{N} \). Show that \( j_2 := \min A_2 \) exists.
   (d) Show that there exists a strictly increasing sequence \((a_n)\) in \( I \) a sequence of natural numbers \((j_n)\) such that if \( a_0 := a \), we have
      i. \( \frac{1}{j_{n+1} + 1} \geq a_{n+1} - a_n > \frac{1}{j_{n+1}} \) for all \( n \geq 0 \)
      ii. if \( c > a \) is such that \( c - a_n > 0 \) and \( |f(x) - f(y)| < \epsilon \text{ on } [a, c] \), then we have \( \frac{1}{j_{n+1} + 1} \geq c - a_n \)
   (e) Define \( L := \lim a_n \) by the bounded monotone convergence theorem, show that \( L = b \).
   (f) Hence, show that \( f \) is uniformly continuous on \([a, b]\).

Remark. The technique used in this question is called Exhaustion Argument because the sequence \((a_n)\) is defined by considering the most optimal objects we can construct.

9. Let \( f : A \to \mathbb{R} \) be a function on some subset. We say that \( f \) is lower semi-continuous \((\text{lsc})\) at \( x \in A \) if for all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( |y - x| > \delta \) with \( y \in A \), then \( f(x) - f(y) < \epsilon \).
   a. Show that \( f \) is lower semi-continuous at \( x \in A \) if and only if sequences \((x_n)\) in \( A \) with \( \lim x_n = x \), we have \( f(x) \leq \lim \inf f(x_n) \).
   b. We say that \( f \) is lower semi-continuous on \( A \) if it is at every point of \( A \). Suppose \( A \) is compact and \( f \) is lower semi-continuous. Show that \( f \) bounded below and minimum is attained, that is,
      \[ \inf\{f(x) : x \in A\} = \min\{f(x) : x \in A\} \]

10. Let \( f : A \to \mathbb{R} \) be a lower semi-continuous function (see the previous question for the definition).
   a. For all \( n \in \mathbb{N} \), define \( f_n(x) := \inf\{f(y) + n|x - y| : y \in X\} \) for all \( x \in A \). Show that \( f_n \) are well-defined continuous functions on \( A \).
   b. Show that there exists an increasing sequence of continuous functions \((g_n : A \to \mathbb{R})\), that is, \( g_n(x) \leq g_{n+1}(x) \) for all \( n \in \mathbb{N} \) and \( x \in A \), such that \( \lim_n g_n(x) = f(x) \) for all \( x \in A \).