MATH 2058 - HW 6 - Solutions

1 (P.110 Q15)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$.

a. Show that f has a limit at $x = 0$.

b. Let $c \neq 0$. Show that f does not have a limit at c using a sequential argument.

Solution.

a. We claim that $\lim_{x\to 0} f(x) = 0$ and proceed to prove it using the definition of functional limits. Let $\epsilon > 0$. Take $\delta := \epsilon$. Suppose $x \in \mathbb{R}$ with $0 < |x| < \delta$. If $x \in \mathbb{Q}$, then it holds that $|f(x) - 0| = |x - 0| < \delta = \epsilon$. If $x \notin \mathbb{Q}$, then it holds that $|f(x) - 0| = |0 - 0| = 0 < \delta = \epsilon$. It follows that $|f(x) - 0| < \epsilon$ for all $x \in \mathbb{R}$ with $0 < |x| < \delta$. By definition, $\lim_{x\to 0} f(x) = 0$

b. Since c is a cluster point(why?), the sequential criteria for limits says that $\lim_{x\to c} f(x)$ exists if and only if there exists $L \in \mathbb{R}$ such that every sequence (x_n) converging to c with $x_n \neq 0$ for all $n \in \mathbb{N}$ satisfies that $\lim f(x_n) = L$. It suffices to find two sequences converging to c whose limits converge differently.
Note that $\mathbb{Q}$ is **dense**. We can pick a sequence (q_n) of rational numbers such that $\lim q_n = c$ (for example, we can pick the sequence with $|q_n - c| < 1/n$ for all $n \in \mathbb{N}$). Note also that $f(q_n) = q_n$ for all $n \in \mathbb{N}$ by definition of f. It follows that $\lim f(q_n) = \lim q_n = c$.
On the other hand $\mathbb{R}\backslash\mathbb{Q}$ is also **dense**. We can pick a sequence (α_n) of irrational numbers such that $\lim \alpha_n = c$. It then follows that $\lim f(\alpha_n) = \lim 0 = 0$. Hence, $\lim f(\alpha_n) = 0 \neq c = \lim f(q_n)$ where (α_n) and (q_n) converges to c. This shows that $\lim_{x\to c} f(x)$ does not exist.

2 (P.116 Q4)**.** Prove the following assertions:

a. The limit $\lim_{x\to 0} \cos(1/x)$ does not exist.

b. The limit $\lim_{x\to 0} x\cos(1/x)$ exists and is equal to 0.

Solution.

a. **Method 1: Sequential Criteria**. We can use this method because 0 is a cluster point of the domain of $f(x) := \cos(1/x)$.
Consider $x_n := \frac{1}{2n\pi}$ and $y_n := \frac{1}{(2n+1)\pi}$ for all $n \in \mathbb{N}$. Then it is clear that $\lim x_n = \lim y_n = 0$ (for instance by Squeeze Theorem). However we have $\cos(1/x_n) = \cos(2n\pi) = 1$ for all $n \in \mathbb{N}$ while $\cos(1/y_n) = \cos((2n+1)\pi) = -1$ for all $n \in \mathbb{N}$. It follows that $\lim \cos(1/x_n) = 1 \neq -1 = \lim \cos(1/y_n)$. It follows from the sequential criteria that $\lim_{x\to 0} \cos(1/x)$ does not exist.

Method 2: Cauchy Criteria. Write $f(x) := \cos(1/x)$ for all $x \neq 0$. Take $\epsilon_0 := 2$. Let $\delta > 0$. Then by the Archimedean property, there exists $n_\delta \in \mathbb{N}$ such that $0 < \frac{1}{2n_\delta \pi}, \frac{1}{(2n_\delta+1)\pi} < \delta$. Now take $x_\delta := 1/2n_0\pi$ and $y_0 = 1/(2n_\delta + 1)\pi$. Then we have

$$|f(x_\delta) - f(y_\delta)| = |1 - (-1)| \geq 2 = \epsilon_0$$

It follows from the negation of Cauchy Criteria that the limit does not exist.

b. We proceed by using the definition of functional limits. Write $g(x) := x\cos(1/x)$ for $x \neq 0$. Let $\epsilon > 0$. Take $\delta := \epsilon$. Now suppose $x \in (-\delta, \delta)\backslash\{0\}$. Then we have

$$|g(x) - 0| = |x\cos(1/x)| = |x||\cos(1/x)| \leq |x| < \delta = \epsilon$$

in which we have used the fact that the image of cosine is bounded by 1. It follows from the definition that $\lim_{x\to 0} \cos(1/x) = 0$

3 (P.129 Q10)**.** Show that the absolute function function $f(x) := |x|$ defined on $\mathbb{R}$ is continuous everywhere on $\mathbb{R}$.

Solution. We proceed using the definition of continuity. Let $c \in \mathbb{R}$. Let $\epsilon > 0$. Take $\delta := \epsilon$. Now suppose $x \in \mathbb{R}$ such that $|x - c| < \delta$. It follows that

$$|f(x) - f(c)| = ||x| - |c|| \leq |x - c| < \delta = \epsilon$$

in which we have used the triangle inequality. It follows that f is continuous at $c \in \mathbb{R}$. Since c is arbitrary, we conclude that f is continuous everywhere.