THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 3 (February 10)

1 The Limit of a Sequence

Definition. A sequence $X = (x_n)$ in $\mathbb{R}$ is said to **converge** to $x \in \mathbb{R}$, or x is said to be a **limit** of (x_n), if for every $\varepsilon > 0$ there exists a natural number $K(\varepsilon)$ such that for all $n \geq K(\varepsilon)$, the terms x_n satisfy $|x_n - x| < \varepsilon$.

Notations: $\lim X = x, \quad \lim(x_n) = x, \quad \lim_n x_n = x, \quad \lim_{n\to\infty} x_n = x.$

Procedure. *To show that* $\lim(x_n) = x$*, we proceed as follow:*

(1) Fix an $\varepsilon > 0$*. (*ε *is arbitrary, but cannot be changed once fixed.)*

(2) Find a useful estimate for $|x_n - x|$*.*

(3) Find $K(\varepsilon) \in \mathbb{N}$ *such that the estimate in (2) is less than* ε *whenever* $n \geq K(\varepsilon)$*.*

(4) Complete the proof.

Example 1. Use the definition to show that $\lim \dfrac{1}{n^2+1} = 0$.

Solution. Let $\varepsilon > 0$ be given. Note that

$$\left|\frac{1}{n^2+1} - 0\right| = \frac{1}{n^2+1} < \frac{1}{n^2} \leq \frac{1}{n} \qquad \text{for } n \in \mathbb{N}.$$

By Archimedean Property, there is $K \in \mathbb{N}$ such that $K > 1/\varepsilon$. Now if $n \geq K$, then $1/n \leq 1/K < \varepsilon$, and thus

$$\left|\frac{1}{n^2+1} - 0\right| \leq \frac{1}{n} < \varepsilon.$$

Hence $\lim 1/(n^2+1) = 0$. ◀

Example 2. Use the definition to show that $\lim(\sqrt{n+1} - \sqrt{n}) = 0$.

Solution. We multiply and divide by $\sqrt{n+1} + \sqrt{n}$ to get

$$0 < \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1})^2 - (\sqrt{n})^2}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n}}.$$

Let $\varepsilon > 0$. By Archimedean Property, there is $K \in \mathbb{N}$ such that $K > 1/\varepsilon^2$. Now if $n \geq K$, we have $1/\sqrt{n} \leq 1/\sqrt{K} < \varepsilon$, and hence

$$\left|\sqrt{n+1} - \sqrt{n}\right| < \frac{1}{\sqrt{n}} < \varepsilon.$$

◀

Definition. If $X = (x_1, x_2, x_3, \ldots, x_n, \ldots)$ is a sequence of real numbers and if m is a given natural number, then the m-**tail** of X is the sequence

$$X_m := (x_{m+n} : n \in \mathbb{N}) = (x_{m+1}, x_{m+2}, \ldots).$$

For example, if $X = (1/n : n \in \mathbb{N})$, then $X_{1997} = (1/1998, 1/1999, \cdots)$.

Theorem. *Let $X = (x_n : n \in \mathbb{N})$ be a sequence of real numbers and let $m \in \mathbb{N}$. Then the m-tail $X_m = (x_{m+n} : n \in \mathbb{N})$ of X converges if and only if X converges. In this case, $\lim X_m = \lim X$.*

Proof. Write $X_m = (y_k : k \in \mathbb{N})$. Then $y_k = x_{k+m}$ for any $k \in \mathbb{N}$.

Assume X converges to x. Then given any $\varepsilon > 0$, there is $K(\varepsilon) \in \mathbb{N}$ with $K(\varepsilon) > m$ such that

$$|x_k - x| < \varepsilon \quad \text{for all } k \geq K(\varepsilon),$$

which implies that

$$|y_k - x| = |x_{k+m} - x| < \varepsilon \quad \text{for all } k \geq K(\varepsilon) - m.$$

By taking $K_m(\varepsilon) = K(\varepsilon) - m$, we conclude that X_m converges to x.

Conversely, assume that X_m converges to x. Then given any $\varepsilon > 0$, there is $K_m(\varepsilon) \in \mathbb{N}$ such that

$$|y_k - x| < \varepsilon \quad \text{for all } k \geq K_m(\varepsilon),$$

which implies that

$$|x_k - x| = |y_{k-m} - x| < \varepsilon \quad \text{for all } k \geq K_m(\varepsilon) + m.$$

By taking $K(\varepsilon) = K_m(\varepsilon) + m$, we conclude that X converges to x.

Therefore, X converges to x if and only if X_m converges to x.

□

Classwork

1. Use the definition to show that $\lim \left(\dfrac{n^2 - n}{2n^2 + 3} \right) = \dfrac{1}{2}$.

2. If $\lim(x_n) = x$ and $x \neq 0$, show that there exists a natural number K such that if $n \geq K$, then $\frac{1}{2}|x| < |x_n| < 2|x|$.