THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 10 (April 14)

Definition. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If there exists a constant $K > 0$ such that

$$|f(x) - f(u)| \leq K|x - u| \qquad \text{for all } x, u \in A, \tag{$*$}$$

then f is said to be a **Lipschitz function** (or to satisfy a **Lipschitz condition**) on A.

Remarks. When A is an interval I, the condition $(*)$ means that the slopes of all line segments joining two points on the graph of $y = f(x)$ over I are bounded by some number K.

Theorem. *If $f : A \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on A.*

Example 1. Show that

(a) $f(x) := x^2$ is a Lipschitz function on $[0, b]$, $b > 0$, but does not satisfy a Lipschitz condition on $[0, \infty)$.

(b) $g(x) := \sqrt{x}$ is uniformly continuous on $[0, 2]$ but not a Lipschitz function on $[0, 2]$.

(c) $g(x) := \sqrt{x}$ is a Lipschitz function on $[a, \infty)$, $a > 0$.

(d) $g(x) := \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Intermediate Value Theorem. *Let I be an interval and let $f\colon I \to \mathbb{R}$ be continuous on I. If $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$, then there exists $c \in I$ between a and b such that $f(c) = k$.*

Preservation of Interval Theorem. *Let I be an interval and let $f\colon I \to \mathbb{R}$ be continuous on I. Then the set $f(I)$ is an interval.*

Classwork

1. Show that if f and g are Lipschitz functions on $[a, b]$, then the product fg is also a Lipschitz function on $[a, b]$. Is it true if $[a, b]$ is replaced by (a, b)?

2. Show that the polynomial $p(x) \coloneqq x^4 + 7x^3 - 9$ has at least two real roots.