Math 2050, quick note of Week 4

1. Convergence and ordering

Preserving of ordering under convergence.

Theorem 1.1. *Suppose x_n and y_n are two sequence of real numbers such that $x_n \leq y_n$ for all n. If $\lim_{n\to+\infty} x_n = x$ and $\lim_{n\to+\infty} y_n = y$, then $x \leq y$.*

A simple consequence is the Squeeze theorem:

Theorem 1.2 (Squeeze theorem)**.** *Suppose x_n, y_n and z_n are sequences of real numbers such that*

$$x_n \leq y_n \leq z_n$$

for all $n \in \mathbb{N}$. If $\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} z_n = L$, then $\{y_n\}$ is convergent with $\lim_{n\to+\infty} y_n = L$.

The upshot: The "closed" inequality will be preserved under convergence.
question: What about the opposite? Namely if the limit lies on some interval, is the tail of the sequence also lies inside it?

Theorem 1.3. *Suppose x_n is a sequence of real number such that $\lim_{n\to+\infty} x_n = x$. If $x \in (a, b)$ for some a, b, then there is $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in (a, b)$.*

One of the application is the following special case:

Theorem 1.4. *Suppose x_n is a sequence of positive real number such that $\lim_{n\to+\infty} \frac{x_{n+1}}{x_n} < 1$, then $x_n \to 0$ as $n \to +\infty$.*

2. Criterion of convergence

We would like to determine the convergence of a particular sequence. By boundedness Theorem, a convergent sequence must be bounded.

Example: $x_n = (-1)^n$ is clearly bounded but divergent.

Question: What extra structure can guarantee the convergence?
We first consider a special type of sequences.

Definition 2.1.
1. *A sequence x_n is said to be increasing if $x_{n+1} \geq x_n$ for all n;*
2. *A sequence x_n is said to be decreasing if $x_{n+1} \leq x_n$ for all n;*
3. *A sequence x_n is said to be monotone if it is either increasing or decreasing.*

In this case, the boundedness Theorem is also a sufficient condition.

Theorem 2.1 (Monotone convergence theorem)**.** *Suppose $\{x_n\}$ is a sequence of real numbers which is monotone, then $\{x_n\}$ is convergent if and only if $\{x_n\}$ is bounded.*

Consider the sequence $x_n = (-1)^n$. Although it is divergent, it is not far from being convergent. Namely, $x_{2n} = 1$ and $x_{2n+1} = -1$ for all n which are both convergent.

We need the concept of sub-sequence.

Definition 2.2. *Given a sequence of integer $n_1 < n_2 < ... < n_k < ...$, the sequence $\{x_{n_k}\}_{k=1}^{\infty}$ is said to be a sub-sequence of the original sequence $\{x_n\}$.*

Theorem 2.2. *Suppose $\{x_n\}$ is a convergent sequence, then any sub-sequence $\{x_{n_k}\}_{k=1}^{\infty}$ is convergent with the same limit.*

Using the terminology, we can state the definition of divergence by the following equivalent form.

Theorem 2.3. *Given a sequence $\{x_n\}$, then the following is equivalent:*

(1) *x_n is NOT convergent to x;*
(2) *$\exists \varepsilon_0 > 0$, and a subsequence $\{x_{n_k}\}$ such that for all k,*

$$|x_{n_k} - x| \geq \varepsilon_0$$

Moreover, the boundedness is almost equivalent to convergence in the following sense.

Theorem 2.4 (Bolzano-Weierstrass Theorem)**.** *Suppose $\{x_n\}$ is a bounded sequence, then there is a convergent subsequence.*

We will give an alternative proof which is different from that in textbook.

Proof. By boundedness, there is a, b such that for all n,

$$a \leq x_n \leq b.$$

For $k = 0$, we denote $I_0 = [a, b]$, $a_0 = a$ and $b_0 = b$. Suppose $[a, \dfrac{a_0 + b_0}{2}]$ contains infinity many x_k, then we choose $a_1 = a_0$, $b_1 = \frac{a_0+b_0}{2}$ otherwise we choose $a_1 = \frac{a_0+b_0}{2}$ and $b_1 = b_0$. Then we define $I_1 = [a_1, b_1]$ and pick $x_{n_1} \in I_1$. This is possible since I_1 contains infinity many elements.

We repeat the same step to obtain a sequence of I_k so that I_k is a sequence of closed, bounded and nested sequence. Moreover, there is $x_{n_k} \in I_k$ and

$$|I_k| = \frac{b-a}{2^k}.$$

By nested interval theorem, we have $\eta \in \cap_{k=1}^{\infty} I_k$. Therefore,

$$|\eta - x_{n_k}| \leq |I_k| = \frac{b-a}{2^k}$$

which implies $x_{n_k} \to \eta$ as $k \to +\infty$. □