Math 2050, quick note of Week 2

1. Density of Rational and Irrational numbers on $\mathbb{R}$

From numerical point of view, we approximate $\sqrt{2}$ by 1.41421356237....Precisely, what we are doing is: finding a sequence of rational number, namely

$$\begin{cases} a_1 = 1; \\ a_2 = 1.4; \\ a_3 = 1.41; \\ a_4 = 1.414... \end{cases} \tag{1.1}$$

so that a_n gets closer and closer to "THE" number $\sqrt{2}$ which is the abstract number obtained from completeness. This suggests a density nature of $\mathbb{Q}$. And here is the general result.

Theorem 1.1 (Density of rational number). *For all $x, y \in \mathbb{R}$ such that $x < y$, we can find $q \in \mathbb{Q}$ such that $q \in (x, y)$.*

Example: We have

$$\sup\{q \in \mathbb{Q} : q^2 < 2, q > 0\} = \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}.$$

(We can think of $\mathbb{R}$ as the minimal completion of $\mathbb{Q}$ so that the "missing hole" is filled.)

And similarly, the irrational number is also dense.

Theorem 1.2 (Density of rational number). *For all $x, y \in \mathbb{R}$ such that $x < y$, we can find $q \notin \mathbb{Q}$ such that $q \in (x, y)$.*

And hence irrational number are also "almost everywhere" inside $\mathbb{R}$.

2. Intervals

For notational convenience, we will use

1. $(a, b) = \{x : a < x < b\}$;
2. $[a, b) = \{x : a \le x < b\}$;
3. $(a, b] = \{x : a < x \le b\}$;
4. $[a, b] = \{x : a \le x \le b\}$;
5. $(a, +\infty) = \{x : a < x\}$;
6. $[a, +\infty) = \{x : a \le x\}$;
7. $(-\infty, b) = \{x : x < b\}$;
8. $(-\infty, b] = \{x : x \le b\}$;
9. $(-\infty, +\infty) = \mathbb{R}$.

Hence, we can rephrase density as "Any non-empty open interval contains element in $\mathbb{Q}$ and $\mathbb{Q}^c$."

Question: How do we determine whether a subset of $\mathbb{R}$ is a interval or not?

Theorem 2.1 (Characterization of Interval). *If S is a non-empty subset of $\mathbb{R}$ such that S contains two distinct real numbers and satisfies the following property:*

$$\text{For any } x, y \in S, \text{ we have } [x, y] \subset S;$$

then S is an interval.

2.1. **Special type of intervals.** For a sequence of interval $\{I_n\}_{n=1}^{\infty}$. We say that the sequence is nested if

$$I_k \subset I_{k-1}$$

for all $k \geq 1$. In particular, the sequence is "decreasing".

Example: $I_n = (0, \frac{1}{n})$, then $\cap_{n=1}^{\infty} I_n = \emptyset$. This is because if $x \in I_n$ for all n, then

$$0 < x < \frac{1}{n}.$$

But this contradicts with the Archimedean property.

Example: $I_n = [0, \frac{1}{n})$, then $\cap_{n=1}^{\infty} I_n = \{0\}$ since for $x \in \cap_{n=1}^{\infty} I_n$, we have for all n that

$$0 \leq x < \frac{1}{n}.$$

Clearly, 0 satisfies the above. And from Archimedean property, positive number fails to satisfies it and hence the assertion holds.

Example: $I_n = [n, +\infty)$, then $\cap_{n=1}^{\infty} I_n = \emptyset$ since for $x \in \cap_{n=1}^{\infty} I_n$, we have for all n that

$$x \geq n$$

which contradicts with the Archimedean property.

The above examples show that for a nested interval to have common intersection, it is necessary that

(a) I_n are bounded;
(b) I_n are closed,

for all n. It turns out to be sufficient as well:

Theorem 2.2 (Nested Interval Theorem). *Suppose $\{I_n = [a_n, b_n]\}_{n=1}^{\infty}$ is a sequence of nested, closed and bounded interval on $\mathbb{R}$, then $\cap_{n=1}^{\infty} I_n$ is non-empty. Moreover, if $\inf\{b_n - a_n\} = 0$, then $\cap_{n=1}^{\infty} I_n$ is a singleton.*

Remark 2.1. For those who are interested in "Axiomatic" construction of $\mathbb{R}$, one can replace the completeness axiom of $\mathbb{R}$ by "Archimedean property and Nested Interval Property". The constructed $\tilde{\mathbb{R}}$ will be identical to the construction using completeness axiom. Google it if you want to know!

Theorem 2.3. $[0, 1]$ *is uncountable.*

Proof. Suppose $[0, 1]$ is countable. That is to say that the set $[0, 1]$ is enumerative:

$$[0, 1] = \{x_n\}_{n=1}^{\infty}.$$

Our goal is to construct some sequence which contradicts with something. We now construct a sequence of interval $\{I_n\}_{n=1}^{\infty}$ which are nested, closed and bounded.

Step 0. We choose $I_0 = [0, 1]$.

Step 1. Considering $x_1 \in [0, 1]$, we choose a subinterval $I_1 \subset I_0$ such that I_0 is closed and $x_1 \notin I_1$. This is possible since x_1 is simply a point!

Step 2. Considering $x_2 \in [0, 1]$. If $x_2 \notin I_1$, then we take $I_2 = I_1$. Otherwise, we find a subinterval $I_2 \subset I_1$ such that I_2 is closed and $x_2 \notin I_2$.

. . . .

Step $k, k > 2$**.** Consider $x_k \in [0, 1]$. If $x_k \notin I_{k-1}$, then we take $I_k = I_{k-1}$. Otherwise, we find a subinterval $I_k \subset I_{k-1}$ such that I_k is closed and $x_k \notin I_k$.

(We are doing each steps ONE BY ONE!)

In this way, $\{I_n\}_{n=1}^{\infty}$ is a sequence of nested interval which are closed and bounded. Hence, Nested Interval Theorem implies $\eta \in \cap_{n=1}^{\infty} I_n \subset I_0 = [0, 1]$. By our assumption, $\eta = x_N$ for some N since $[0, 1] = \{x_n\}_{n=1}^{\infty}$. This implies

$$x_N \in I_N \cap I_N^c$$

which is impossible.

□