Series

Def: Let $(x_n) \subset \mathbb{R}$. Define $s_n := \sum_{k=1}^{n} x_k$.

We say $\sum_{n=1}^{\infty} x_n$ is convergent if s_n is convergent.

Important Examples:

i) $\sum r^k$ converges if $0 \leq r < 1$ &
diverges if $r > 1$.

ii) $\sum \frac{1}{k^p}$ converges if $p > 1$ &
diverges if $p \leq 1$.

Exercise 1: Let $\{(x_n)\} \subset \mathbb{R}$. Set

$$r := \limsup |x_n|^{\frac{1}{n}}.$$

Show that

(i) If $r<1$, then $\sum_{n=1}^{\infty} x_n$ is absolutely convergent.

(ii) If $r>1$, then $\sum_{n=1}^{\infty} x_n$ is divergent.

Proof: (i) Let $\alpha \in (r,1)$. Then $\exists N \in \mathbb{N}$ s.t for $n \geq N$,

$$|x_n|^{\frac{1}{n}} < \alpha. \Rightarrow |x_n| < \alpha^n.$$

Hence, for $m > n \geq N$, we have

$$\left| \sum_{k=1}^{m} |x_k| - \sum_{k=1}^{n} |x_k| \right| \leq |x_{n+1}| + \cdots + |x_m|$$

$$\leq \alpha^{n+1} + \cdots + \alpha^{m} \leq \frac{\alpha^{n+1}}{1-\alpha} \leq \frac{\alpha^{N+1}}{1-\alpha}.$$

Let $\varepsilon > 0$. Since $\lim_n \alpha^n = 0$, $\exists N' \in \mathbb{N}$,

s.t. $\frac{\alpha^{N'+1}}{1-\alpha} < \varepsilon$.

This proves the sequence $\{\sum_{k=1}^{n} |x_k|\}$ is

Cauchy and hence convergent. (Let $m > n \geq \max\{N, N'\}$).

Hence $\sum_{k=1}^{\infty} x_k$ converges absolutely.

(ii) Let $\alpha \in (1, r)$. Then $\forall n, \exists m > n$ s.t.

$$|x_m|^{\frac{1}{m}} > \alpha \Rightarrow |x_m| > \alpha^m > 1.$$

By the n-th term test, $\sum_{n=1}^{\infty} x_n$ diverges.

$\square$

<u>Question</u>:

What about $r = 1$?

Compare $\{\frac{2}{n}\}$ & $\{\frac{2}{n^2}\}$.