THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050A Mathematical Analysis I (Fall 2021)
Suggested Solution of Homework 5

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1. (2 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $S = \{x \in \mathbb{R} : f(x) = 0\}$. Show that S is closed in the sense that if $x_n \in S$ and $x_n \to x$, then $x \in S$.

Solution: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $x_n \in S$ for all $n \in \mathbb{N}$ and $x_n \to x$. Then $f(x_n) = 0$ for all $n \in \mathbb{N}$. Since f is continuous, we have that

$$f(x) = \lim_{t \to x} f(t) = \lim_{n \to \infty} f(x_n) = 0.$$

Therefore $x \in S$.

2. (2 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous functions such that

$$f(m2^{-n}) = m2^{-n}$$

for all $m \in \mathbb{Z}, n \in \mathbb{N}$. Show that $f(x) = x$ for all $x \in \mathbb{R}$.

Solution: Let $x \in \mathbb{R}$. For any $n \in \mathbb{N}$, we can find $m_n \in \mathbb{Z}$ such that

$$x2^n \leq m_n < x2^n + 1.$$

Let $x_n = m_n 2^{-n}$. Then

$$x \leq x_n < x + 2^{-n}.$$

By squeeze theorem, we have that $\lim_{n\to\infty} x_n = x$. Since $m_n \in \mathbb{Z}$ and $n \in \mathbb{N}$, we have that $f(x_n) = x_n$. By continuity of f, we can conclude that

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = x.$$

3. (2 points) Let $I = [0, \pi/2]$ and $f : I \to \mathbb{R}$ be a function given by $f(x) = \sup\{x^2, \cos x\}$ for $x \in I$. Show that there is $x_0 \in I$ such that $f(x_0) = \min\{f(x) : x \in I\}$. Moreover, $x_0^2 = \cos x_0$.

Solution: Since I is a closed interval, by Max-Min Theorem, to prove f attains a minimum in I, it suffices to show that f is continuous on I.

Let $h, g : I \to \mathbb{R}$ be two continuous functions.

We prove the claim that $f(x) = \sup\{h(x), g(x)\}$ is continuous at any $c \in I$.

(1) If $h(c) = g(c)$, for any $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that $|h(x) - h(c)| < \epsilon$ for $|x - c| < \delta_1$ and $|g(x) - h(c)| < \epsilon$ for $|x - c| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. For $|x - c| < \delta$, we have that

$$|f(x) - f(c)| \leq \sup\{|h(x) - h(c)|, |g(x) - h(c)|\} < \epsilon.$$

It follows that f is continuous at c.

(2) If $h(c) \neq g(c)$, without loss of generality, we can assume that $h(c) = f(c) > g(c)$. For $\epsilon = \frac{h(c)-g(c)}{2}$, there exist $\delta_1, \delta_2 > 0$ such that $|h(x) - h(c)| < \epsilon$ for $|x - c| < \delta_1$ and $|g(x) - g(c)| < \epsilon$ for $|x - c| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. For $|x - c| < \delta$, we have that

$$g(x) < \frac{h(c) + g(c)}{2} < h(x).$$

Hence $f(x) = h(x)$ for $|x - c| < \delta$. Therefore, f is continuous at c.

Alternatively, we can also prove the continuity of f by the fact that

$$f = \frac{1}{2}(h + g + |h - g|).$$

From the claim above, we can find $x_0 \in I$ such that $f(x_0) = \min\{f(x) : x \in I\}$.

Let $F(x) = x^2 - \cos(x)$. We have that F is continuous, strictly increasing with $F(0) = -1 < 0$ and $F(\frac{\pi}{2}) = \frac{\pi^2}{4} > 0$. Then there exists $b \in I$ such that $F(b) = 0$. Suppose $x_0 \neq b$. Let $x' = \frac{x_0+b}{2} \in I$.

(1) If $x_0 > b$, then $x_0 > x' > b$. By monotonicity of F, $F(x_0) > F(x') > 0$. Then $f(x_0) = x_0^2 > x'^2 = f(x')$, contradicting the minimality of $f(x_0)$.

(2) If $x_0 < b$, then $x_0 < x' < b$. By monotonicity of F, $F(x_0) < F(x') < 0$. Then $f(x_0) = \cos(x_0) > \cos(x') = f(x')$, also a contradiction.

Therefore, $F(x_0) = 0$, which implies that $x_0^2 = \cos(x_0)$.

4. (2 points) Show that $f(x) = x^{-1}$ on $(a, +\infty)$ is uniformly continuous if $a > 0$. Is the result still true if $a = 0$? Give your reasoning.

Solution: Let $a > 0$. For any $\epsilon > 0$, let $\delta = a^2\epsilon$. Then for $x_1, x_2 > a$, whenever $|x_1 - x_2| < \delta$, we have that

$$\left|x_1^{-1} - x_2^{-1}\right| = \frac{|x_1 - x_2|}{|x_1 x_2|} \leq a^{-2}\,|x_1 - x_2| < \epsilon.$$

Hence x^{-1} is uniformly continuous on $(a, +\infty)$.

For $a = 0$, let $x_n = \frac{1}{n}$ and $u_n = \frac{1}{n+1}$ be two seuqences in $(0, +\infty)$.

Then $\lim_{n\to\infty} |x_n - u_n| = 0$ and $\lim_{n\to\infty} |x_n^{-1} - u_n^{-1}| = 1 > 0$. Hence x^{-1} is not uniformly continuous on $(0, +\infty)$.

5. (2 points) Suppose $f : [0,1] \to \mathbb{R}$ is a function such that for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq \Lambda |x - y|^{1/2}$$

for some $\Lambda > 0$. Show that f is uniformly continuous. Is the converse also true? Give your reasoning.

Solution: For any $\epsilon > 0$, let $\delta = \Lambda^{-2}\epsilon^2$. Then for $x_1, x_2 \in [0,1]$, whenever $|x_1 - x_2| < \delta$, we have that

$$|f(x_1) - f(x_2)| \leq \Lambda \left|x_1 - x_2\right|^{\frac{1}{2}} < \epsilon.$$

Hence f is uniformly continuous on $[0,1]$.

Conversely, $f(x) = x^{\frac{1}{4}}$ is a continuous function on $[0, +\infty)$, thus is uniformly continuous on $[0,1]$. However,

$$\frac{|f(x) - f(0)|}{|x - 0|^{\frac{1}{2}}} = x^{-\frac{1}{4}} \to \infty \quad \text{as} \quad x \to 0.$$

Therefore the converse is not true.