THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2040A (First Term,2021-22)
Linear Algebra II
Tutorial 2

Sec. 1.5

16 We will prove the contrapositive of the statement.

If S is linearly dependent, there exists distinct $u_1, \dots, u_n \in S$ and $a_1, \dots, a_n \in \mathsf{F}$ not all zero such that $\sum_{i=1}^{n} a_i u_i = \overrightarrow{0}$. Then $S' := \{u_1, \dots, u_n\} \subset S$ is finite and by above we have S' being linearly dependent.

Now if there exists a finite subset S' of S which is linearly dependent, there exists distinct $u_1, \dots, u_n \in S'$ and $a_1, \dots, a_n \in \mathsf{F}$ not all zero such that $\sum_{i=1}^{n} a_i u_i = \overrightarrow{0}$. Since $S' \subset S$, $u_1, \dots, u_n$ are distinct vectors in S. Therefore $\sum_{i=1}^{n} a_i u_i = \overrightarrow{0}$ is a nontrivial representation of $\overrightarrow{0}$ by vectors in S and hence S is linearly dependent.

17 Assume the columns of M to be $\{v_i\}_{i=1}^{n}$ and

$v_i = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ii} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, where $a_{ii} \neq 0$. Let $c_i \in R$ for $i = 1, 2, \cdots, n$. If $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$, we have

$$\begin{cases} c_1 a_{11} + c_2 a_{12} + \cdots + c_n a_{1n} = 0 \\ \qquad\quad c_2 a_{22} + \cdots + c_n a_{2n} = 0 \\ \qquad\qquad \vdots \\ \qquad\qquad\qquad c_n a_{nn} = 0. \end{cases} \tag{1}$$

Consider the last equation, we have $c_n = 0$ since $a_{nn} \neq 0$. Assume for all $i \geq k$, $c_i = 0$. When $i \geq k-1$, from the equation

$$c_{k-1} a_{k-1,k-1} + c_k a_{k-1,k} + \cdots + c_n a_{k-1,n} = 0 \tag{2}$$

and the fact that $a_{k-1,k-1} \neq 0$, $c_k = c_{k+1} = \cdots = c_n = 0$, we obtain that $c_{k-1} = 0$. Then we conclude that $c_i = 0$ for $i = 1, 2, \cdots, n$. By definition, the columns of M are linearly independent.

20 Suppose $a, b \in \mathbb{R}$ such that $af + bg = \overrightarrow{0}$. Then for all $t \in \mathbb{R}$ we have $ae^{rt} + be^{st} = 0$. In particular, we have $0 = ae^{r0} + be^{s0} = a + b$ and $0 = ae^{r1} + be^{s1} = ae^r + be^s$. Therefore $0 = e^s(a+b) - (ae^r + be^s) = a(e^s - e^r)$. Since $r \neq s$, $e^s - e^r \neq 0$. Hence $a = 0$. Similarly, we have $b = 0$. Therefore a, b must be zero and f, g are linearly independent.