THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2040A (First Term, 2021-22)
Linear Algebra II
Tutorial 4

Sec. 1.6

12 Q: Let u, v and w be distinct vectors of a vector space V. Show that if $\{u, v, w\}$ is a basis for V, then $\{u+v+w, v+w, w\}$ is also a basis for V.

Sol: Suppose a, b, c are scalars such that $a(u+v+w)+b(v+w)+cw=\vec{0}$. Then

$$au+(a+b)v+(a+b+c)w=\vec{0}.$$

By linear independence of $\{u, v, w\}$, $a = a+b = a+b+c = 0$. Then, $a = b = c = 0$. Therefore, $\{u+v+w, v+w, w\}$ is linearly independent.
Because $\{u, v, w\}$ is a basis for V, V is of dimension 3 and by Corollary 2 in Sec. 1.6, $\{u+v+w, v+w, w\}$ is a basis for V.
(**Alternatively**, as $V = \operatorname{span}\{u, v, w\}$, $\forall x \in V$, $\exists$ scalars a, b, c such that

$$x = au+bv+cw = a(u+v+w)+(b-a)(v+w)+(c-b-a)w.$$

Thus, $V = \operatorname{span}\{u+v+w, v+w, w\}$.
Finally, as $\{u+v+w, v+w, w\}$ spans V and is linearly independent, it is a basis for V.)

30 Q: Let

$$V = \mathsf{M}_{2\times 2}(F), \quad W_1 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in V : a, b, c \in F \right\},$$

and

$$W_2 = \left\{ \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \in V : a, b \in F \right\}.$$

Prove that W_1 and W_2 are subspaces of V, and find the dimensions of W_1, W_2, $W_1 + W_2$ and $W_1 \cap W_2$.

Sol: Clearly,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -0 & 0 \end{pmatrix} \in W_1 \cap W_2.$$

Let $A, A' \in W_1$ and $\alpha \in F$. Then

$$A = \begin{pmatrix} a & b \\ c & a \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} a' & b' \\ c' & a' \end{pmatrix}$$

for some $a, b, c, a', b', c' \in F$. We have

$$A - A' = \begin{pmatrix} a-a' & b-b' \\ c-c' & a-a' \end{pmatrix} \in W_1 \quad \text{and} \quad \alpha A = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha a \end{pmatrix} \in W_1.$$

Hence, W_1 is a subspace of V.
Let $B, B' \in W_2$ and $\beta \in F$. Then

$$B = \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \quad \text{and} \quad B' = \begin{pmatrix} 0 & a' \\ -a' & b' \end{pmatrix}$$

for some $a, b, a', b' \in F$. We have

$$B - B' = \begin{pmatrix} 0 & a - a' \\ -(a - a') & b - b' \end{pmatrix} \in W_2 \quad \text{and} \quad \beta A = \begin{pmatrix} 0 & \beta a \\ -\beta a & \beta b \end{pmatrix} \in W_2.$$

Hence, W_2 is a subspace of V.
We can see that (we omit the proof)

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

is a basis for W_1 while

$$\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis for W_2. So $\dim(W_1) = 3$ and $\dim(W_2) = 2$.
Define

$$W_3 = \left\{ \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \in V : a \in F \right\}.$$

Clearly, $W_3 \subset W_1 \cap W_2$. Suppose $C \in W_1 \cap W_2$. Then

$$C = \begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{pmatrix} 0 & d \\ -d & e \end{pmatrix}$$

for some $a, b, c, d, e \in F$. Then $a = e = 0$, $b = d = -c$, whence $C \in W_3$. We have $W_1 \cap W_2 = W_3$. Note that (we omit the proof)

$$\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

is a basis for W_3. So $\dim(W_1 \cap W_2) = \dim(W_3) = 1$.
Finally, applying the equality in 29 (a), we have

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = 3 + 2 - 1 = 4.$$