THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2040A (First Term, 2021-22)
Linear Algebra II
Tutorial 3

Optional Part

Sec. 1.6

29 Q: (a) Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace $W_1 + W_2$ is finite-dimensional, and $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$. *Hint:* Start with a basis $\{u_1, ..., u_k\}$ for $W_1 \cap W_2$ and extend this set to a basis $\{u_1, ..., u_k, v_1, ..., v_m\}$ for W_1 and to a basis $\{u_1, ..., u_k, w_1, ..., w_p\}$ for W_2.

(b) Let W_1 and W_2 be finite-dimensional subspaces of a vector space V, and let $V = W_1 + W_2$. Deduce that V is the direct sum of W_1 and W_2 if and only if $\dim(V) = \dim(W_1) + \dim(W_2)$.

Sol: (a) We pick a basis $\{u_1, ..., u_k\}$ for $W_1 \cap W_2$. In particular, $\{u_1, ..., u_k\}$ is linearly independent. By Corollary 2 in Sec. 1.6, we can extend $\{u_1, ..., u_k\}$ to a basis $\{u_1, ..., u_k, v_1, ..., v_m\}$ for W_1 and to a basis $\{u_1, ..., u_k, w_1, ..., w_p\}$ for W_2.
Then we claim that $\{u_1, ..., u_k, v_1, ..., v_m, w_1, ..., w_p\}$ is a basis for $W_1 + W_2$.
Indeed,

$$\begin{aligned} W_1 + W_2 &= \operatorname{span}(\{u_1, ..., u_k, v_1, ..., v_m\}) + \operatorname{span}(\{u_1, ..., u_k, w_1, ..., w_p\}) \\ &= \operatorname{span}(\{u_1, ..., u_k, v_1, ..., v_m\} \cup \{u_1, ..., u_k, w_1, ..., w_p\}) \\ &= \operatorname{span}(\{u_1, ..., u_k, v_1, ..., v_m, w_1, ..., w_p\}). \end{aligned}$$

(See Exercise Q14 in Sec. 1.4.)
On the other hand, suppose $a_1, ..., a_k, b_1, ..., b_m, c_1, ..., c_p$ are scalars such that

$$a_1u_1 + \cdots + a_ku_k + b_1v_1 + \cdots + b_mv_m + c_1w_1 + \cdots c_pw_p = \vec{0}. \tag{1}$$

Rearranging, we have

$$b_1v_1 + \cdots + b_mv_m = -(a_1u_1 + \cdots + a_ku_k + c_1w_1 + \cdots c_pw_p) \in W_1 \cap W_2$$

and hence $\exists$ scalars $d_1, ..., d_k$ such that $b_1v_1 + \cdots + b_mv_m = -(d_1u_1 + \cdots + d_ku_k)$.
Rearranging again,

$$d_1u_1 + \cdots + d_ku_k + b_1v_1 + \cdots + b_mv_m = \vec{0}.$$

By linear independence of $\{u_1, ..., u_k, v_1, ..., v_m\}$, we have, in particular, $b_1 = \cdots = b_m = 0$. Then (1) reduces to

$$a_1u_1 + \cdots + a_ku_k + c_1w_1 + \cdots c_pw_p = \vec{0}.$$

We apply the same trick again, making use of linear independence of $\{u_1, ..., u_k, w_1, ..., w_p\}$ to show that $a_1 = \cdots = a_k = c_1 = \cdots = c_p = 0$. $\{u_1, ..., u_k, v_1, ..., v_m, w_1, ..., w_p\}$ is then linearly independent and hence a basis for $W_1 + W_2$. In particular, $W_1 + W_2$ is finite-dimensional.

Finally, we see that

$$\begin{aligned}\dim(W_1 + W_2) =& k + m + p = (k + m) + (k + p) - k \\ =& \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).\end{aligned}$$

(b) ($\Rightarrow$) Suppose $V = W_1 \oplus W_2$. Then $W_1 \cap W_2 = \{\vec{0}\}$ and by (a)

$$\begin{aligned}\dim(V) &= \dim(W_1 + W_2) \\ &= \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) \\ &= \dim(W_1) + \dim(W_2)\end{aligned}$$

since $\dim(W_1 \cap W_2) = 0$.

($\Leftarrow$) Suppose $\dim(V) = \dim(W_1) + \dim(W_2)$. Then by (a), $\dim(W_1 \cap W_2) = \dim(V) - \dim(W_1) - \dim(W_2) = 0$, whence $W_1 \cap W_2 = \{\vec{0}\}$. By hypothesis, $V = W_1 + W_2$. Thus, $V = W_1 \oplus W_2$.

33 Q: (a) Let W_1 and W_2 be subspaces of a vector space V such that $V = W_1 \oplus W_2$. If β_1 and β_2 are bases for W_1 and W_2, respectively, show that $\beta_1 \cap \beta_2 = \emptyset$ and $\beta_1 \cap \beta_2$ is a basis for V.

(b) Conversely, let β_1 and β_2 be disjoint bases for subspaces W_1 and W_2, respectively, of a vector space V. Prove that if $\beta_1 \cup \beta_2$ is a basis for V, then $V = W_1 \oplus W_2$.

Sol: (a) We first show that $\beta_1 \cup \beta_2$ is a basis for V.

Indeed, fix $v \in V$. $\exists w_1 \in W_1$ and $w_2 \in W_2$ such that $v = w_1 + w_2$. Then $\exists$ scalars $a_1, ..., a_m, b_1, ..., b_n$, vectors $u_1, ..., u_m \in \beta_1$ and $v_1, ..., v_n \in \beta_2$ such that

$$v = a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n \in \operatorname{span}(\beta_1 \cup \beta_2).$$

Hence, $V = \operatorname{span}(\beta_1 \cup \beta_2)$.

Suppose $a_1, ..., a_n$ are scalars and $u_1, ..., u_n \in \beta_1 \cup \beta_2$ such that

$$a_1u_1 + \cdots + a_nu_n = \vec{0}.$$

By relabelling the u_i's if necessary, assume $u_1, ..., u_k \in \beta_1$ and $u_{k+1}, ..., u_n \in \beta_2$. Then

$$a_1u_1 + \cdots + a_ku_k = -(a_{k+1}u_{k+1} + \cdots + a_nu_n) \in W_1 \cap W_2 = \{\vec{0}\},$$

whence $a_1u_1 + \cdots + a_ku_k = (-a_{k+1})u_{k+1} + \cdots + (-a_n)u_n = \vec{0}$. By linear independence of both β_1 and β_2, $a_1 = \cdots = a_n = 0$. Hence, $\beta_1 \cup \beta_2$ is linearly independent.

Therefore, $\beta_1 \cup \beta_2$ is a basis for V.

We next prove $\beta_1 \cap \beta_2 = \emptyset$ by contradiction.

Assume the contrary that $\beta_1 \cap \beta_2 \neq \emptyset$. Then we can pick an element $u \in \beta_1 \cap \beta_2$. In particular, $u \in W_1 \cap W_2 = \{\vec{0}\}$. If forces that $1 \cdot u = u = \vec{0}$. But $1 \neq 0$. It violates the linear independence of β_1. Therefore, $\beta_1 \cap \beta_2 = \emptyset$.

(b) Suppose $v \in V$. Then $\exists$ scalars $a_1, ..., a_n$ and vectors $u_1, ..., u_n \in \beta_1 \cup \beta_2$ such that $v = \sum_{i=1}^n a_i u_i$. By relabelling the u_i's if necessary, assume $u_1, ..., u_k \in \beta_1$ and $u_{k+1}, ..., u_n \in \beta_2$. Then $v = \sum_{i=1}^k a_i u_i + \sum_{i=k+1}^n a_i u_i \in W_1 + W_2$. Hence, $V = W_1 + W_2$.

Suppose $w \in W_1 \cap W_2$. $\exists$ scalars $a_1, ..., a_m, b_1, ..., b_n$, distinct vectors $u_1, ..., u_m \in \beta_1$ and distinct vectors $v_1, ..., v_n \in \beta_2$ such that $w = \sum_{i=1}^m a_i u_i = \sum_{j=1}^n b_j v_j$. Then we have $\sum_{i=1}^m a_i u_i + \sum_{j=1}^n (-b_j) v_j = \vec{0}$. Note that $\beta_1 \cap \beta_2 = \emptyset$. It implies that $u_1, ..., u_m, v_1, ..., v_n$ are all distinct. Then by linear independence of $\beta_1 \cup \beta_2$, $a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0$, whence $w = \vec{0}$. It implies that $W_1 \cap W_2 = \{\vec{0}\}$.

All in all, $V = W_1 \oplus W_2$.

34 Q: (a) Prove that if W_1 is any subspace of a finite-dimensional vector space V, then there exists a subspace W_2 of V such that $V = W_1 \oplus W_2$.

(b) Let $V = \mathbb{R}^2$ and $W_1 = \{(a_1, 0) : a_1 \in \mathbb{R}\}$. Give examples of two different subspaces W_2 and W_2' such that $V = W_1 \oplus W_2$ and $V = W_1 \oplus W_2'$.

Sol: (a) Choose a basis B_1 for W_1. In particular, B_1 is linearly independent. By Corollary 2 in Sec. 1.6, B_1 can be extended to a basis B for V. Let $B_2 = B \backslash B_1$ and define $W_2 = \operatorname{span} B_2$. We claim that $V = W_1 \oplus W_2$.

Indeed, $W_1 + W_2 \subset V$ clearly. Consider any $v \in V$. $\exists$ scalars $a_1, ..., a_n$ and $u_1, ..., u_n \in B$ such that $v = \sum_{i=1}^n a_i u_i$. Note that $B = B_1 \cup B_2$. Without loss of generality, we assume $u_1, ..., u_k \in B_1$ and $u_{k+1}, ..., u_n \in B_2$ (otherwise we just relabel the u_i's and a_i's). Then

$$v = \sum_{i=1}^k a_i u_i + \sum_{i=k+1}^n a_i u_i \in W_1 + W_2.$$

Therefore, $V = W_1 + W_2$.

On the other hand, obviously $\vec{0} \in W_1 \cap W_2$. Suppose $w \in W_1 \cap W_2$. Then $\exists$ scalars $a_1, ..., a_m, b_1, ..., b_n$, distinct vectors $u_1, ..., u_m \in B_1$ and distinct vectors $v_1, ..., v_n \in B_2$ such that

$$w = \sum_{i=1}^m a_i u_i = \sum_{j=1}^n b_j v_j.$$

Since $B_1 \cap B_2 = \emptyset$, $u_1, ..., u_m, v_1, ..., v_n$ are distinct. Rearranging the above equation yields

$$a_1 u_1 + \cdots + a_m u_m + (-b_1) v_1 + \cdots + (-b_n) v_n = \vec{0}.$$

Then by linear independence of B, $a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0$, whence $w = \vec{0}$. Thus, $W_1 \cap W_2 = \{\vec{0}\}$.

To conclude, $V = W_1 \oplus W_2$.

(b) Let $W_2 = \{(0, a_2) : a_2 \in \mathbb{R}\}$ and $W_2' = \{(a, a) : a \in \mathbb{R}\}$. They are subspaces of V.

$\forall a, b \in \mathbb{R}$, $(a, b) = (a, 0) + (0, b) \in W_1 + W_2$ and $(a, b) = (a - b, 0) + (b, b) \in W_1 + W_2'$. Hence, $V = W_1 + W_2 = W_1 + W_2'$. It remains to show that $W_1 \cap W_2 = W_1 \cap W_2' = \{(0, 0)\}$. Suppose $v \in W_1 \cap W_2$. Then $v = (a_1, 0) = (0, a_2)$ for some $a_1, a_2 \in \mathbb{R}$ and hence $v = (0, 0)$. Thus $W_1 \cap W_2 = \{(0, 0)\}$. Suppose $w \in W_1 \cap W_2'$. Then $v = (a_1, 0) = (a, a)$ for some $a_1, a \in \mathbb{R}$, whence $v = (0, 0)$. Thus $W_1 \cap W_2' = \{(0, 0)\}$.

To conclude, $V = W_1 \oplus W_2 = W_1 \oplus W_2'$.