

Topic#8

Invertibility & Isomorphism

goal:

If $n = \dim(V) < \infty$ and $m = \dim(W) < \infty$

then, T is bijective if and only if $m = n$ and $[T]_{\alpha}^{\beta}$ is non-singular.

Note: $A \in M_{n \times n}(F)$ is non-singular $\iff \det(A) \neq 0 \iff$
 A is invertible $\iff \dots$

Def. $T \in \mathcal{L}(V, W)$. T is **invertible** if there exists a function

$$U : W \rightarrow V$$

such that

$$TU = I_W \text{ and } UT = I_V.$$

Remark(1) T is invertible **iff** T is bijective.

Pf:

\Rightarrow (a) T is onto. Indeed, let $y \in W$,

then, $T(U(y)) = TU(y) = I_W(y) = y$

i.e. $\exists U(y) \in V$ s.t. $T(U(y)) = y$.

(b) $T : V \rightarrow W$ is one-to-one. Indeed, let $T(x) = T(y)$, $x, y \in V$,

then $U(T(x)) = U(T(y))$ i.e. $UT(x) = UT(y)$ i.e. $I_V(x) = I_V(y)$

i.e. $x = y$ □

$\Leftarrow U \stackrel{\text{def}}{=} T^{-1}$

Remark(2) If T is invertible then U is unique, given $U = T^{-1}$.

Thm. If $T \in \mathcal{L}(V, W)$ is invertible

then $T^{-1} : W \rightarrow V$ is linear, so $T^{-1}(\cdot) : W \rightarrow V$ is linear.

Proof. Let $y_1, y_2 \in W, a \in \mathbb{F}$.

$\because T$ is invertible $\therefore T$ is bijective, T^{-1} exists uniquely.

$\therefore \exists! x_1 = T^{-1}(y_1), x_2 = T^{-1}(y_2) \in V$, s.t.

$$y_1 = T(x_1), y_2 = T(x_2).$$

Then,

$$\begin{aligned} & T^{-1}(a_1 y_1 + a_2 y_2) \\ &= T^{-1}(a_1 T(x_1) + a_2 T(x_2)) \\ &= T^{-1}(T(a_1 x_1 + a_2 x_2)) \\ &= a_1 x_1 + a_2 x_2 \\ &= a_1 T^{-1}(y_1) + a_2 T^{-1}(y_2) \end{aligned}$$



Remarks:

(1) If $T : V \rightarrow W$ is invertible then $T^{-1} : W \rightarrow V$ is invertible and $(T^{-1})^{-1} = T$.

(2) If $T : V \rightarrow W$ and $U : W \rightarrow Z$ are invertible, then $UT : V \rightarrow Z$ is invertible and $(UT)^{-1} = T^{-1}U^{-1}$.

e.g.: Let $A \in M_{n \times n}(\mathbb{F})$, and

$$x \in \mathbb{F}^n \mapsto L_A(x) = Ax \in \mathbb{F}^n$$

Then,

$L_A \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$ is invertible **iff** $A \in M_{n \times n}(\mathbb{F})$ is invertible.

In this case,

$$(L_A)^{-1} = L_{A^{-1}}.$$



Lemma. Let $T \in \mathcal{L}(V, W)$ be invertible. Then

$$\dim(V) < \infty \text{ iff } \dim(W) < \infty.$$

In this case, $\dim(V) = \dim(W) < \infty$.

Proof. “ \Rightarrow ” Let $\dim(V) < \infty$. Let β be a finite basis for V .

$$W \stackrel{T \text{ is onto}}{=} R(T) = \text{span}(T(\beta)) \quad \therefore \dim(W) \leq n < \infty.$$

“ \Leftarrow ” Let $\dim(W) < \infty$,

apply $T^{-1} \in \mathcal{L}(W, V)$ to show $\dim(V) < \infty$. □

Let $\dim(V) = n < \infty$, $\beta = \{v_1, \dots, v_n\}$ a basis for V . Then
 $W = \text{span}(\{T(v_1), \dots, T(v_n)\})$

Claim: $T(\beta) = \{T(v_1), \dots, T(v_n)\}$ is l. indep.
($\because T$ is one-to-one).

If so, $T(\beta)$ is a basis for W . $\#T(\beta) = n \therefore \dim(W) = n = \dim(V)$.

Proof of claim:

Let $\sum_{i=1}^n a_i T(v_i) = 0_v$ for $a_1, \dots, a_n \in F$.

To show: $a_1 = \dots = a_n = 0$.

$$\because 0 = \sum_{i=1}^n a_i T(v_i) = T(\sum_{i=1}^n a_i v_i) \quad (\because T \in \mathcal{L})$$

And T is one-to-one.

$$\therefore \sum_{i=1}^n a_i v_i = 0$$

$\because \beta = \{v_1, \dots, v_n\}$ is l.indep.

$$\therefore a_1 = \dots = a_n = 0.$$

Thm. Let $T \in \mathcal{L}(V, W)$, where V, W are finite-dimensional with ordered bases β, γ , respectively. Then

T is invertible **iff** $[T]_{\beta}^{\gamma}$ is invertible.

Moreover,

$$[T^{-1}]_{\gamma}^{\beta} = ([T]_{\beta}^{\gamma})^{-1}.$$

Proof. “ \Rightarrow ” Assume: T is invertible.

First, $\dim(V) = \dim(W)$ by lemma. Let $n = \dim(V) = \dim(W)$.
By $T^{-1}T = 1_V$,

$$I_{n \times n} = [I_V]_{\beta}^{\beta} = [T^{-1}T]_{\beta}^{\beta} = [T^{-1}]_{\gamma}^{\beta} [T]_{\beta}^{\gamma}.$$

Similarly, by $TT^{-1} = I_W$,

$$I_{n \times n} = [I_W]_{\gamma}^{\gamma} = [TT^{-1}]_{\gamma}^{\gamma} = [T]_{\beta}^{\gamma} [T^{-1}]_{\gamma}^{\beta}.$$

$\therefore [T]_{\beta}^{\gamma}$ is invertible, and $([T]_{\beta}^{\gamma})^{-1} = [T^{-1}]_{\gamma}^{\beta}$. □

“ \Leftarrow ” Assume: $A \stackrel{\text{def}}{=} [T]_{\beta}^{\gamma}$ is invertible (of finite size)

To show $T \in \mathcal{L}(V, W)$ is invertible. It suffices to show T is one to one.

Let $v_1, v_2 \in V$, and $T(v_1) = T(v_2)$.

$$\Rightarrow [T(v_1)]_{\gamma} = [T(v_2)]_{\gamma}$$

$$\Rightarrow [T]_{\beta}^{\gamma}[v_1]_{\beta} = [T]_{\beta}^{\gamma}[v_2]_{\beta}$$

$$\Rightarrow [v_1]_{\beta} = [v_2]_{\beta} \quad (\because [T]_{\beta}^{\gamma} \text{ is invertible})$$

$$\Rightarrow v_1 = v_2.$$



Remark:

$$\begin{array}{ccc} V & \begin{array}{c} \xrightarrow{T} \\ \xleftarrow{T^{-1}} \end{array} & W \\ \downarrow [\cdot]_{\beta} & & \downarrow [\cdot]_{\gamma} \\ \mathbb{F}^{\dim(V)} & \begin{array}{c} \xrightarrow{[T]_{\beta}^{\gamma}} \\ \xleftarrow{[T^{-1}]_{\gamma}^{\beta}} \end{array} & \mathbb{F}^{\dim(W)} \end{array}$$

T is invertible $\Leftrightarrow [T]_{\beta}^{\gamma}$ is invertible

$$V \xrightarrow{\exists U \in \mathcal{L}(W, V)} W$$

$$\begin{array}{|l} \bullet \sum_{i=1}^n B_{i1} v_i \\ \bullet \sum_{i=1}^n B_{i2} v_i \\ \vdots \\ \bullet \sum_{i=1}^n B_{in} v_i \end{array} \leftarrow \begin{array}{|l} \bullet w_1 \\ \bullet w_2 \\ \vdots \\ \bullet w_n \end{array}$$

such that $U(w_j) = \sum_{i=1}^n B_{ij} v_i, j = 1, \dots, n$.

By def.: $[B_{ij}]_{n \times n} = [U]_{\gamma}^{\beta}$.



Corollary. $T \in \mathcal{L}(V)$, where $\dim(V) < \infty$ and β is an ordered basis for V . Then,

T is invertible **iff** $[T]_\beta$ is invertible.

Moreover, in this case,

$$[T^{-1}]_\beta = ([T]_\beta)^{-1}.$$

$$\begin{array}{ccc} V & \begin{array}{c} \xrightarrow{T} \\ \xleftarrow{T^{-1}} \end{array} & V \\ \downarrow [\cdot]_\beta & & \downarrow [\cdot]_\gamma \\ \mathbb{F}^n & \begin{array}{c} \xrightarrow{[T]_\beta} \\ \xleftarrow{[T^{-1}]_\beta} \end{array} & \mathbb{F}^n \end{array}$$

Def. Let V, W : v.s. Then, V is **isomorphic** to W if there is an invertible $T \in \mathcal{L}(V, W)$.

In this case, T is called an **isomorphism** from V onto W .

Thm. Let V, W be finite-dimensional v.s.. Then,

V is isomorphic to W **iff** $\dim(V) = \dim(W)$.

Proof. “ \Rightarrow ” Assume: V is isomorphic to W .

$\therefore \exists$ an isomorphism $T \in \mathcal{L}(V, W)$

$\therefore T$ is invertible \therefore By the previous lemma, $\dim(V) = \dim(W)$. \square

“ \Leftarrow ” Assume: $\dim(V) = \dim(W) \stackrel{\text{def}}{=} n < \infty$. Let

$\beta = \{v_1, \dots, v_n\}$: basis for V

$\gamma = \{w_1, \dots, w_n\}$: basis for W

Then, $\exists! T \in \mathcal{L}(V, W)$ such that $T(v_i) = w_i, i = 1, \dots, n$.

Then $R(T) = \text{span}(T(\beta)) = \text{span}(\gamma) = W$

$\therefore T$ is onto, hence one-to-one ($\dim(V) = \dim(W) < \infty$)

$\therefore T$ is bijective. So $T \in \mathcal{L}(V, W)$ is invertible. So T is an isomorphism.

$\therefore V$ is isomorphic to W . $\square \square$

Corollary. Let V be a v.s. over \mathbb{F} . Then

V is isomorphic to \mathbb{F}^n **iff** $\dim(V) = n$.

e.g. set $\dim(V)=n$ and β is an o.b. for V .

Write the standard representation of V under β as $[\cdot]_{\beta} = \phi_{\beta}$.

Then take any $v \in V$, see $[v]_{\beta} \in \mathbb{F}^n$ where $[v]_{\beta}$ is β -coordinate of $v \in V$.

The $[\cdot]_{\beta} : V \rightarrow \mathbb{F}^n$ is isomorphism.

Thm. Let V, W be finite-dimensional v.s. over \mathbb{F} with $\dim(V) = n$, $\dim(W) = m$, and ordered bases β, γ , resp.

$$\begin{array}{ccc}
 V & \begin{array}{c} \xrightarrow{T \in \mathcal{L}(V, W)} \\ \xleftarrow{T^{-1}} \end{array} & W \\
 \downarrow [\cdot]_{\beta} & & \downarrow [\cdot]_{\gamma} \\
 \mathbb{F}^n & \begin{array}{c} \xrightarrow{[T]_{\beta}^{\gamma} \in M_{m \times n}(\mathbb{F})} \\ \xleftarrow{\quad} \end{array} & \mathbb{F}^m
 \end{array}$$

Then, the mapping

$$\begin{aligned}
 \Phi : \mathcal{L}(V, W) &\rightarrow M_{m \times n}(\mathbb{F}) \\
 T &\mapsto \Phi(T) = [T]_{\beta}^{\gamma}
 \end{aligned}$$

is an isomorphism.

(i.e. $\mathcal{L}(V, W)$ is isomorphic to $M_{m \times n}(\mathbb{F})$).

This tells: (1) $\mathcal{L}(V, W)$ is finite-dimensional.

(2) $\dim(\mathcal{L}(V, W)) = \dim(M_{m \times n}(\mathbb{F})) = mn$.

Proof. 1°. Φ is linear.

2° Φ is one-to-one.

$\Phi(T_1) = \Phi(T_2)$ i.e. $[T_1]_{\beta}^{\gamma} = [T_2]_{\beta}^{\gamma}$ to show: $T_1 = T_2$

take $v \in V$, $[T_1(v)]_{\gamma} = [T_1]_{\beta}^{\gamma}[v]_{\beta}$, $[T_2(v)]_{\gamma} = [T_2]_{\beta}^{\gamma}[v]_{\beta}$

$\therefore [T_1(v)]_{\gamma} = [T_2(v)]_{\gamma} \therefore T_1(v) = T_2(v)$.

3°. Φ is onto. let $A = (a_{ij})_{m \times n} \in M_{m \times n}(\mathbb{F})$.

To show: $\exists T \in \mathcal{L}(V, W)$ s.t. $A = \Phi(T) = [T]_{\beta}^{\gamma}$.

Indeed, $\beta = \{v_1, \dots, v_n\}$ o.b. for V and $\gamma = \{w_1, \dots, w_m\}$ o.b. for W .

Then, $\exists! T \in \mathcal{L}(V, W)$ such that $T(v_j) = \sum_{i=1}^m a_{ij} w_i$, $1 \leq j \leq n$.

$\therefore A = [T]_{\beta}^{\gamma} = \Phi(T)$, i.e. T is onto.

Def. Let V be a v.s. over \mathbb{F} with $\dim(V) = n$, and β be an ordered basis for V . The map

$$\begin{aligned}\Phi_\beta : V &\rightarrow \mathbb{F}^n \\ v &\mapsto \Phi_\beta(v) \stackrel{\text{def.}}{=} [v]_\beta\end{aligned}$$

is called the **standard representation** of V w.r.t. β .

Note: Φ_β is an isomorphism from V to \mathbb{F}^n .

