

Topic#3

Span & linear (in-)dependence

$(V, +, \cdot)$: v.s. over \mathbb{F}

Def.

-

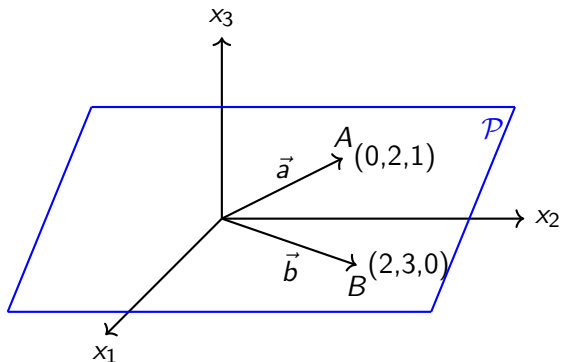
$$\sum_{i=1}^m a_i v_i = a_1 v_1 + \cdots + a_m v_m \in V$$

is called a **linear combination** of $v_1, \dots, v_m \in V$ with combination coefficients $a_1, \dots, a_m \in \mathbb{F}$.

- Let $\emptyset \neq S \subset V$.

$\text{span}(S) \stackrel{\text{def}}{=} \text{a set of all possible linear combinations of vectors in } S$

$$= \left\{ \sum_{i=1}^m a_i v_i : \text{each } a_i \in \mathbb{F}, \text{each } v_i \in S, 1 \leq i \leq m, m = 1, 2, \dots \right\}$$



\mathcal{P} consists of all vectors of the form

$$s\vec{a} + t\vec{b} \text{ (linear combination), } s, t \in \mathbb{R}$$

$$\mathcal{P} = \text{span}\{\vec{a}, \vec{b}\} \text{ (span of } \{\vec{a}, \vec{b}\})$$

Note:

(1) Any linear combination contains **only finite** many terms.

(2) Even if **S is infinite**, we can still define $\text{span}(S)$ well.

(3) Convention: if **$S = \emptyset$** , then $\text{span}(\emptyset) = \{0\}$.

An example: $f, f_1, f_2 \in \mathbb{P}_3(\mathbb{R})$: How to find $a, b \in \mathbb{R}$ s.t.
 $f = af_1 + bf_2$?

(generally, $v \in V = \sum_{i=1}^m a_i v_i$, how to determine a_i ?)

Claim: it is equivalent to solve a linear system $Ax = b$
 $f = 2x^3 - 2x^2 + 12x - 6$, $f_1 = x^3 - 2x^2 - 5x - 3$,
 $f_2 = 3x^3 - 5x^2 - 4x - 9$.

Plug the three equations to $f = af_1 + bf_2$.

Get: $2 = a + 3b$, $-2 = -2a - 5b$, $12 = -5a - 4b$, $-6 = -3a - 9b$.

$\Rightarrow \exists! a = -4 \in \mathbb{R}, b = 2 \in \mathbb{R}$



Prop. $(V, +, \cdot)$: v.s. over \mathbb{F} . $S \subset V$. Then

(1). $\text{span}(S)$ is a subspace of V , and

(2). $\text{span}(S)$ is the smallest subspace of V containing S in the sense that if W is a subspace with $W \supset S$, then

$$W \supset \text{span}(S).$$

namely, any subspace containing S must contain $\text{span}(S)$.

Proof. $S = \emptyset$, by convention, $\text{span}(\emptyset) = \{0\}$ is a subspace of V
Of course is a smallest subspace of V containing \emptyset .

Assume $V \supset S \neq \emptyset$.

(1) to show $\text{span}(S)$ is a subspace of V . Indeed,

(a) $\text{span}(S) \subset V$.

In fact, take $v \in \text{span}(S)$.

By def of span, $v = \sum_{i=1}^m a_i v_i$ with $v_i \in S$, $a_i \in \mathbb{F}$.

$\because S \subset V \therefore$ all $v_i \in V$.

\therefore Since V is a v.s., $v = \sum_{i=1}^m a_i v_i \in V$. □

(b) $0 \in \text{span}(S)$

Indeed, $S \neq \emptyset$, $\exists v \in S$. $0 = 0v \in \text{span}(S)$

since LHS is zero of V , 0 at RHS is zero scalar of \mathbb{F} and $v \in S$. □

(c) Let $u, v \in \text{span}(S)$, to show $u + v \in \text{span}(S)$. Indeed,

$\because u, v \in \text{span}(S)$

$$u = \sum_{i=1}^m a_i u_i, \quad v = \sum_{i=1}^n b_i v_i \quad \text{with } a_i, b_i \in \mathbb{F}, u_i, v_i \in S,$$

$$\text{then } u + v = (a_1 u_1 + \cdots + a_m v_m) + (b_1 v_1 + \cdots + b_n v_n)$$

is still a linear combination of vectors

$$u_1, \cdots, u_m, v_1, \cdots, v_n \in \text{span}(S) \text{ and } a_1, \cdots, a_m, b_1, \cdots, b_n \in \mathbb{F}.$$

$\therefore u + v \in \text{span}(S)$.

(d) Let $a \in \mathbb{F}$, $u \in \text{span}(S)$. Let $u = \sum_{i=1}^m a_i u_i$ for $a_i \in \mathbb{F}$, $u_i \in S$.

Then for $a \in \mathbb{F}$,

$$au = a \left(\sum_{i=1}^m a_i u_i \right) = \sum_{i=1}^m (aa_i) u_i \in \text{span}(S) \quad \text{since } aa_i \in \mathbb{F}, u_i \in S$$

It is a linear combination.

$\therefore \text{span}(S)$ is a subspace of V .



(2) Assume W is a subspace of V and $W \supset S$,
to show: $W \supset \text{span}(S)$.

Take $v \in \text{span}(S)$. Then,

$$v = \sum_{i=1}^m a_i v_i, \quad a_i \in \mathbb{F}, v_i \in S.$$

Since $S \subset W$ all $v_i \in W$

$\therefore W$ is a subspace of V i.e. $W \subset V$ is a v.s., each $v_i \in W$.

$$\therefore v = \sum_{i=1}^m a_i v_i \in W.$$

$\therefore \text{span}(S) \subset W$.



Def. $(V, +, \cdot)$: v.s. over \mathbb{F} . $S \subset V$. We say that S **spans** V if

$$V = \text{span}(S) \text{ for some } S \subset V$$

e.g. * $\mathbb{F}^n = \text{span}\{e_1, \dots, e_n\}$,

where $e_i = (0, \dots, 0, \underset{\substack{\uparrow \\ i^{\text{th}}}}{1}, 0, \dots)$

* $P_n(\mathbb{F}) = \text{span}(\{1, x, x^2, \dots, x^n\})$,

$P(\mathbb{F}) = \text{span}(\{1, x, x^2, \dots\})$,
infinite

* $M_{m \times n}(\mathbb{F}) = \text{span}(\{E_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\})$

where E_{ij} is the matrix with all zero entries except 1 at i^{th} row and j^{th} column.

Basic question: V : v.s. over \mathbb{F} :

(1). Does V have a finite spanning set?

(2). If so, can one find a finite spanning set with the min size?
(linearly (in)dependence)

$(V, +, \cdot)$: v.s. over \mathbb{F}

Def. $S \subset V$ is **linearly dependent** if \exists **distinct** $v_1, \dots, v_m \in S$ and $a_1, \dots, a_m \in \mathbb{F}$ (**not all zero**), s.t.

$$a_1 v_1 + \dots + a_m v_m = 0.$$

Otherwise $S \subset V$ is **linearly independent**.

Remarks:

- (1) $\emptyset \subset V$ is l. indep.; Any l. dep. subset of V must be non-empty.
- (2) If $0 \in S \subset V$, then S is l. dep. ($\because 1 \cdot 0 = 0$)
- (3) $S = \{v\}$ is l. indep. $\Leftrightarrow v \neq 0$.

More observations. Let $S_1 \subset S_2 \subset V$, then

(a) $\text{span}(S_1) \subset \text{span}(S_2)$

(b) if S_1 l. dep. then S_2 l. dep.

(S_1 l.dep. $\stackrel{\text{def}}{\implies} \exists$ distinct $v_1, \dots, v_m \in S_1 \subset S_2$
and $a_1, \dots, a_m \in \mathbb{F}$ (Not all zero) s.t. $a_1 v_1 + \dots + a_m v_m = 0$).

(c) If $V = \text{span}(S_1)$ then $V = \text{span}(S_2)$

Lemma. Let $S \subset V$.

(1). S is l. indep **iff** any finite subset of S is l. indep.

Proof: " \Rightarrow " Otherwise, \dots

" \Leftarrow " Otherwise, S is l.dep., then by def., \exists distinct $v_1, \dots, v_m \in S$ and $a_1, \dots, a_m \in \mathbb{F}$ (not all zero) s.t. $a_1 v_1 + \dots + a_m v_m = 0$).

Def $S_1 \stackrel{\text{def}}{=} \{v_1, \dots, v_m\} \subset S$ contradiction with S_1 is l.indep. \square

(2). Let $S = \{v_1, v_2, \dots, v_n\}$ be a finite subset of V . Then, the following three are equivalent:

(a). S is l. indep.

(b). If $\sum_{i=1}^n a_i v_i = 0$ ($a_i \in F$) then $a_1 = \dots = a_n = 0$.

(c). If $v = \sum_{i=1}^n a_i v_i \in \text{span}(S)$ ($a_i \in F$) then a_1, \dots, a_n are unique.

Proof:

(a) \Leftrightarrow (b):

" \Rightarrow " Note: when S is l.indep., v_1, \dots, v_n are distinct. Let

$\sum_{i=1}^m a_i v_i = 0 (a_i \in \mathbb{F})$, to show: $a_1 = \dots = a_n = 0$.

Indeed, otherwise, not all a_i are zero.

$\therefore S = \{v_1, \dots, v_n\}$ is l.dep. by def.. Contradiction!

" \Leftarrow " Otherwise, S is linearly dependent.

(b) \Leftrightarrow (c):

" \Leftarrow " Let $\sum_{i=1}^n a_i v_i = 0$. Note: $\sum_{i=1}^n 0 v_i = 0 = \sum_{i=1}^n a_i v_i$.

By (c), i.e. by uniqueness of a_i , $a_1 = \dots = a_n = 0$.

" \Rightarrow " Let $v = \sum_{i=1}^n a_i v_i \in \text{span}(S)$, to show: a_1, \dots, a_n are unique.

Indeed, let $v = \sum_{i=1}^n a_i v_i = \sum_{i=1}^n b_i v_i, (b_i \in \mathbb{F})$

$\therefore \sum_{i=1}^n (a_i - b_i) v_i = 0$.

By (b), $a_i - b_i = 0$ for each i , i.e. $a_i = b_i, 1 \leq i \leq n$. □.

Thinking:

(1). $\text{span}(\{v \neq 0\}) = V$, otherwise $\not\subset V, \dots$

(2). $V = \text{span}(V)$, kick away some vectors of V without changing span.

Prop. $(V, +, \cdot)$: v.s. over \mathbb{F} . $S \subset V$ is l. dep. Then

$$\exists v \in S \text{ s.t. } \text{span}(S) = \text{span}(S \setminus \{v\}).$$

i.e. if S is l.dep., then one can remove at least one vector in S without changing its span.

Proof. S l. dep $\Rightarrow \exists$ distinct $v_1, \dots, v_m \in S$ & $a_1, \dots, a_m \in \mathbb{F}$ (not all zero) s.t. $a_1 v_1 + \dots + a_m v_m = 0$. For instance $a_1 \neq 0$, then

$$v_1 = -\frac{a_2}{a_1} v_2 - \dots - \frac{a_m}{a_1} v_m \text{ with } -\frac{a_2}{a_1}, \dots, -\frac{a_m}{a_1} \in \mathbb{F}$$

Then choose $v = v_1$, then

$$\text{span}(S) \subset \text{span}(S \setminus \{v\})$$

Because: “ \supset ”: $S \setminus \{v\} \subset S (\because v \in S)$

“ \subset ”: Let $u \in \text{span}(S)$, then

$$u = b_1 u_1 + \cdots + b_n u_n, b_1, \cdots, b_n \in \mathbb{F}, u_1, \cdots, u_n \in S.$$

In case, some of u_i is $v = v_1$, then one can replace such u_i by $u_i \in \text{span}(\{v_2, \cdots, v_n\})$ where $\{v_2, \cdots, v_n\} \subset S \setminus \{v\}$.

Then, $u \in \text{span}(S \setminus \{v\})$.



Prop. $(V, +, \cdot)$: v.s. over \mathbb{F} . $S \subset V$ is l. indep., $v \in V \setminus S$.
(i.e. $v \notin S, v \in V$). Then $S \cup \{v\}$ is l. dep. **iff** $v \in \text{span}(S)$.

Proof. " \Rightarrow " Assume: $S \cup \{v\}$ is l. dep., to show: $v \in \text{span}(S)$.
Indeed, by def, \exists distinct $u_1, \dots, u_m \in S \cup \{v\}$ and
 $a_1, \dots, a_m \in \mathbb{F}$ (not all zero)

$$\text{s.t. } a_1 u_1 + \dots + a_m u_m = 0. \quad (*)$$

Claim: At least one of u_i should be v .

(otherwise, no u_i is v , it means all $u_1, \dots, u_m \neq v$, then
 $v_1, \dots, v_m \in S$. Note: S is l.indep. then $(*)$ is a contradiction)

□

For instance, assume $u_1 = v$ & $a_1 \neq 0$

$$\therefore v = v_1 = \left(-\frac{a_2}{a_1}\right)v_2 + \cdots + \left(-\frac{a_m}{a_1}\right)v_m \in \text{span}(S)$$

\uparrow \uparrow
 $\in S$ $\in S$

(Note: $\because v_2, \dots, v_m \in S \cup \{v\}$ are distinct with $v_1 = v$
 $\therefore v_2, \dots, v_m \in S$.)

“ \Leftarrow ” Let $v \in \text{span}(S)$ to show: $S \cup \{v\}$ is linearly dep.
Indeed, $\because v \in \text{span}(S)$

$$\therefore v = \sum_{i=1}^m a_i v_i, v_i \in S, a_i \in \mathbb{F}.$$

Namely, $a_1 v_1 + \cdots + a_m v_m + (-1)v = 0$

- (i) $v_1, \dots, v_m, v \in S \cup \{v\}$ distinct ($\because v \notin S$).
- (ii) $a_1, \dots, a_m, -1 (\neq 0)$: not all zero

By def, $S \cup \{v\}$ is l. dep. □