MATH2040A Homework 4 suggested answer

Compulsory Part

Q2.1.4.

Solution: Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} \in M_{2\times 3}(F)$, $\alpha \in F$. Then

$$\begin{aligned} T(A+B) &= T\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{pmatrix} \\ &= \begin{pmatrix} 2(a_{11}-b_{11})-(a_{12}-b_{12}) & (a_{13}+b_{13})+2(a_{12}+b_{12}) \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2a_{11}-a_{12} & a_{13}+2a_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2b_{11}-b_{12} & b_{13}+2b_{12} \\ 0 & 0 \end{pmatrix} \\ &= T(A)+T(B) \end{aligned}$$

and

$$\begin{aligned} T(\alpha A) &= T\begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \end{pmatrix} \\ &= \begin{pmatrix} 2\alpha a_{11}-\alpha a_{12} & \alpha a_{13}+2\alpha a_{12} \\ 0 & 0 \end{pmatrix} \\ &= \alpha\begin{pmatrix} 2a_{11}-a_{12} & a_{13}+2a_{12} \\ 0 & 0 \end{pmatrix} \\ &= \alpha T(A) \end{aligned}$$

As $A, B \in M_{2\times 3}(F)$ and $\alpha \in F$ are arbitrary, T is a linear transformation.
The kernel of T is then

$$\begin{aligned} \mathsf{N}(T) &= \{\, A \in M_{2\times 3}(F) \mid T(A) = 0 \,\} \\ &= \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \;\middle|\; a_{ij} \in F, i \in \{1,2\}, j \in \{1,2,3\}, \begin{pmatrix} 2a_{11}-a_{12} & a_{13}+2a_{12} \\ 0 & 0 \end{pmatrix} = 0 \right\} \\ &= \left\{ \begin{pmatrix} a_{11} & 2a_{11} & -4a_{11} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \;\middle|\; a_{ij} \in F, i \in \{1,2\}, j \in \{1,2,3\} \right\} \\ &= \mathrm{Span}\left(\begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \end{aligned}$$

It is easy to verify that $\left\{ \begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$ forms a basis of $\mathsf{N}(T)$, and nullity $T = \dim \mathsf{N}(T) = 4$. Similarly the range is

$$\begin{aligned} \mathsf{R}(T) &= \{\, T(A) \mid A \in M_{2\times 3}(F) \,\} \\ &= \left\{ \begin{pmatrix} 2a_{11}-a_{12} & a_{13}+2a_{12} \\ 0 & 0 \end{pmatrix} \;\middle|\; a_{11}, a_{12}, a_{13} \in F \right\} \\ &= \mathrm{Span}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \end{aligned}$$

It is easy to verify that $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ forms a basis of $\mathsf{R}(T)$, and $\operatorname{rank} T = \dim \mathsf{R}(T) = 2$

As $\dim M_{2\times 3}(F) = 2 \times 3 = 6$, we have $\operatorname{nullity} T + \operatorname{rank} T = \dim M_{2\times 3}(F)$. So the dimension theorem is verified for T.

Since $\operatorname{nullity} T = 4 \neq 0$, T is not one-to-one. Since $\operatorname{rank} T = 2 \neq 4 = \dim M_{2\times 2}(F)$, T is not onto.

Q2.1.5.

Solution: Let $\alpha \in F$, $p(x) = a_0 + a_1 x + a_2 x^2, q(x) = b_0 + b_1 x + b_2 x^2 \in P_2(F)$ where $a_0, a_1, a_2, b_0, b_1, b_2 \in F$. Then $T(p(x) + q(x)) = x(p(x) + q(x)) + (p(x) + q(x))' = xp(x) + p'(x) + xq(x) + q'(x) = T(p(x)) + T(q(x))$, and $T(\alpha p(x)) = x(\alpha p(x)) + (\alpha p(x))' = \alpha(p(x) + p'(x)) = \alpha T(p(x))$.

As $p(x), q(x), \alpha$ are arbitrary, T is a linear transformation.

The kernel of T is

$$\begin{aligned} \mathsf{N}(T) &= \{\, p(x) \in P_2(F) \mid T(p(x)) = 0 \,\} \\ &= \left\{\, a_0 + a_1 x + a_2 x^2 \;\middle|\; a_0, a_1, a_2 \in F, a_1 + (a_0 + 2a_2)x + a_1 x^2 + a_2 x^3 = 0 \,\right\} \\ &= \left\{\, a_0 + a_1 x + a_2 x^2 \;\middle|\; a_0, a_1, a_2 \in F, a_0 = a_1 = a_2 = 0 \,\right\} \\ &= \{0\} \end{aligned}$$

So a basis of $\mathsf{N}(T)$ is $\emptyset$, and $\operatorname{nullity} T = \dim\{0\} = 0$.

Similarly, the range of T is

$$\begin{aligned} \mathsf{R}(T) &= \{\, T(p(x)) \mid p(x) \in P_2(F) \,\} \\ &= \left\{\, a_1 + (a_0 + 2a_2)x + a_1 x^2 + a_2 x^3 \;\middle|\; a_0, a_1, a_2 \in F \,\right\} \\ &= \operatorname{Span}\left(\left\{\, 1 + x^2, x, 2x + x^3 \,\right\} \right) \end{aligned}$$

It is easy to see that $\left\{\, 1 + x^2, x, 2x + x^3 \,\right\}$ forms a basis of $\mathsf{R}(T)$, so $\operatorname{rank} T = \dim \mathsf{R}(T) = 3$.

As $\dim P_2(F) = 3$, we have $\operatorname{nullity} T + \operatorname{rank} T = \dim P_2(F)$. So the dimension theorem is verified for T.

Since $\mathsf{N}(T) = \{0\}$, T is one-to-one. Since $\operatorname{rank} T = 3 \neq 4 = \dim P_3(F)$, T is not onto.

Q2.1.14.

Solution:

(a) Suppose T is one-to-one. Let $S \subseteq T$ be an linearly independent set. If $S = \emptyset$, $T(S) = \emptyset$ is linearly independent. So we may assume that $S \neq \emptyset$.

Let $T(v_1), \ldots, T(v_n) \in T(S)$ be distinct with $v_1, \ldots, v_n \in S$ where $n \in \mathbb{Z}^+$. Let $a_1, \ldots, a_n \in F$ be scalars such that $\sum_{i=1}^n a_i T(v_i) = 0$. So $T(\sum_{i=1}^n a_i v_i) = 0$. As T is one-to-one, $\sum_{i=1}^n a_i v_i = 0$. As $v_1, \ldots, v_n \in S$, $\{v_1, \ldots, v_n\}$ is linearly independent. Thus $a_1 = \ldots = a_n = 0$. Since this is the only solution, $\{T(a_1), \ldots, T(a_n)\}$ is linearly independent.

As $T(v_1), \ldots, T(v_n) \in T(S)$ and $n \in \mathbb{Z}^+$ are arbitrary, $T(S)$ is linearly independent.

As S is arbitrary, T maps linearly independent sets to linearly independent sets.

Suppose on the other hand that T maps linearly independents sets in V to linearly independent sets in W. Let $v \in \mathsf{N}(T)$. Then $T(v) = 0$. As $\{T(v)\} = \{0\}$ is linearly dependent, $\{v\}$ cannot be linearly independent, for otherwise $T(\{v\}) = \{T(v)\} = \{0\}$ is also linearly independent. Since $\{x\}$ is always linearly independent for $x \in V \setminus \{0\}$, we have $v = 0$. As $v \in \mathsf{N}(T)$ is arbitrary, $\mathsf{N}(T) \subseteq \{0\}$. Trivially $\{0\} \subseteq \mathsf{N}(T)$, so $\mathsf{N}(T) = \{0\}$ and thus T is one-to-one.

(b) Suppose S is linearly independent. By the previous part, $T(S)$, the image of S under T is linearly independent.

Suppose on the other hand that $T(S)$ is linearly independent. The proposition is trivial if $S = \emptyset$, so we may assume that $S \neq \emptyset$.

Let $v_1, \ldots, v_n \in S$ be distinct with $n \in \mathbb{Z}^+$. Let $a_1, \ldots, a_n \in F$ be scalars such that $\sum_{i=1}^n a_i v_i = 0$. Then $0 = T(0) = T(\sum_{i=1}^n a_i v_i) = \sum_{i=1}^n a_i T(v_i)$. As T is one-to-one, $T(v_1), \ldots, T(v_n)$ are distinct. Since $\{T(v_1), \ldots, T(v_n)\} \subseteq T(S)$ and $T(S)$ is linearly independent, $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent and so $a_1 = \ldots = a_n = 0$. Thus $\{v_1, \ldots, v_n\}$ is linearly independent.

As $v_1, \ldots, v_n \in S$ and $n \in \mathbb{Z}^+$ are arbitrary, S is linearly independent.

(c) Since β is a basis and T is one-to-one, by the previous part $T(\beta)$ is linearly independent. So to show that $T(\beta)$ is a basis, it suffices to show that $\operatorname{Span}(T(\beta)) = W$.

Since $T(\beta) \subseteq W$, $\operatorname{Span}(T(\beta)) \subseteq W$.

Let $w \in W$. As T is onto, there exists $v \in V$ such that $w = T(v)$. As $\beta = \{v_1, \ldots, v_n\}$ is a basis of V, there exists scalars $a_1, \ldots, a_n \in F$ such that $v = \sum_{i=1}^n a_i v_i$. So $w = T(v) = T(\sum_{i=1}^n a_i v_i) = \sum_{i=1}^n a_i T(v_i) \in \operatorname{Span}(T(\beta))$. As $w \in W$ is arbitrary, $W \subseteq \operatorname{Span}(T(\beta))$ and so $W = \operatorname{Span}(T(\beta))$.

As $T(\beta)$ is linearly independent and spans W, $T(\beta)$ is a basis of W.

Note

Do not assume that S is a finite set. The proposition holds even when S is an infinite set. Similarly, do not assume that V is finite dimensional and use the dimension theorem.

Please note that the dimension theorem presented in lecture note 06 requires the the domain to be finite dimensional. If you want to use it on (possibly) infinite dimensional spaces, *you may need to prove it first.* Notes for Q1.6.21 in homework 2 are also applicable here.

Note that for part (b) in proving the "if" part you would need to emphasize that the images are distinct, which is due to the injectivity of T (and possibly the only part where the injectivity is used). For a simple example why this is necessary, consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ which maps $(x, y) \in \mathbb{R}^2$ to $(x, 0)$. The map is clearly not one-to-one, but for the linear dependent set $S = \{(1, 0), (1, 1), (1, 2)\} \subseteq \mathbb{R}^2$ we have $T(S) = \{(1, 0)\}$ which is linearly independent.

Note that part (c) is just the lemma on page 6 of lecture note 8.

Q2.1.17.

Solution: The two results are simple consequences of the dimension theorem, which states that $\operatorname{nullity} T + \operatorname{rank} T = \dim V$.

(a) Since V, W are finite dimensional and $\mathsf{N}(T) \subseteq V$, $\dim V, \dim W, \operatorname{nullity} T$ are all finite nonnegative. Also, $\mathsf{R}(T) \subseteq W$.

By dimension theorem, $\dim \mathsf{R}(T) = \operatorname{rank} T = \dim V - \operatorname{nullity} T \leq \dim V < \dim W$. So $\mathsf{R}(T) \neq W$, and thus T is not onto.

(b) Since V, W are finite dimensional and $\mathsf{R}(T) \subseteq W$, $\dim V, \dim W, \operatorname{rank} T$ are all finite. Also, as $\mathsf{R}(T) \subseteq W$, $\operatorname{rank} T \leq \dim W$.

By dimension theorem, $\dim \mathsf{N}(T) = \operatorname{nullity} T = \dim V - \operatorname{rank} T \geq \dim V - \dim W > 0$. So $\mathsf{N}(T) \neq \{0\}$, and thus T is not one-to-one.

Note

You can also prove by contradiction with passing bases between the domain and the codomain.

Q2.1.22.

Solution: Let $a = T(1,0,0), b = T(0,1,0), c = T(0,0,1) \in \mathbb{R}$. Since for $(x,y,z) \in \mathbb{R}^3$ we have $(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$, we also have $T(x,y,z) = T(x(1,0,0) + y(0,1,0) + z(0,0,1)) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1) = ax + by + cz$.

The general result for linear $T : F^n \to F^m$ is as follows:

> there exists scalars $a_{ij} \in F$ for $i \in \{1,\ldots,m\}$, $j \in \{1,\ldots,n\}$ such that $T(x_1,\ldots,x_n) = (\sum_{j=1}^n a_{1j}x_j, \ldots, \sum_{j=1}^n a_{mj}x_j)$ for $(x_1,\ldots,x_n) \in F^n$.

This generalization includes the cases for $T : F^n \to F$ with $m = 1$, where the proposition becomes

> there exists scalars $a_1,\ldots,a_n \in F$ such that $T(x_1,\ldots,x_n) = \sum_{i=1}^n a_i x_i$ for $(x_1,\ldots,x_n) \in F^n$

The proof is similar to the original proposition.

For each $j \in \{1,\ldots,n\}$ let $e_j \in F^n$ be the vector which the jth entry is 1 and all other entries are 0, and $a_{1j},\ldots,a_{mj} \in F$ such that $T(e_j) = (a_{1j},\ldots,a_{mj}) \in F^m$. Such a_{ij} is well-defined as $\{ E_1,\ldots,E_m \}$ forms a basis (the natural basis) of F^m where $E_i \in F^m$ is defined similarly as $e_j \in F^n$.

As for each $x = (x_1,\ldots,x_n) \in F^n$ we have $x = \sum_{j=1}^n x_j e_j$, we also have $T(x) = T(\sum_{j=1}^n x_j e_j) = \sum_{j=1}^n x_j T(e_j) = \sum_{j=1}^n x_j(a_{1j},\ldots,a_{mj}) = (\sum_{j=1}^n a_{1j}x_j, \ldots, \sum_{j=1}^n a_{mj}x_j)$.

Note

Writing the vectors in matrices it is easier to see what this proposition means:

> For linear transformation $T : F^n \to F^m$ there exists a matrix $A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix} \in F^{m\times n}$ such that $T(x) = Ax$ when elements in F^n, F^m are regarded as column vectors. By Theorem 2.6, such A is unique.

Similar proposition also holds if the spaces are not F^n and F^m but generic (finite dimensional) vector spaces with the same scalar field. In this case, the scalars depends on the choice of representations of vectors.

Please note that the map T is already given. You are asked to show the existence of such a, b, c (or a_{ij}), *not* to verify that the map $(x,y,z) \mapsto ax + by + cz$ is linear.

You can also prove the proposition by Theorem 2.6. For brevity we show only the case $T : F^n \to F^m$.

Solution: Let $a_{ij} \in F$ for $i \in \{1,\ldots,m\}$, $j \in \{1,\ldots,n\}$ be defined such that $T(e_i) = (a_{1i},\ldots,a_{mi})$, where $\{e_i \mid i \in \{1,\ldots,n\}\}$ is the natural basis of F^n. We define $U(x) = (\sum_{j=1}^n a_{1j}x_j, \ldots, \sum_{j=1}^n a_{mj}x_j)$ for $x = (x_1,\ldots,x_n) = \sum_{i=1}^n x_i e_i \in F^n$. The map is well-defined and linear as the decomposition by the basis $\{e_i \mid i \in \{1,\ldots,n\}\}$ is unique. It suffices to show that $T = U$.

By Theorem 2.6, it suffices to compare the values at some basis. By definition, for each $i \in \{1,\ldots,n\}$ we have $T(e_i) = (a_{1i},\ldots,a_{mi}) = (\sum_{j=1}^n a_{1j}\delta_{ij}, \ldots, \sum_{j=1}^n a_{mj}\delta_{ij}) = U(e_i)$ where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. As $\{e_i \mid i \in \{1,\ldots,n\}\}$ is a basis of F^n, we have $T = U$.

Therefore there exists scalars $a_{ij} \in F$ for $i \in \{1,\ldots,m\}$, $j \in \{1,\ldots,n\}$ such that $T(x_1,\ldots,x_n) = \sum_{i=1}^n a_i x_i$ for $(x_1,\ldots,x_n) \in F^n$.

Note

Please note how the behavior of a linear map is controlled by its images on a basis.

Q2.1.28.

Solution:

(a) Since $T(0) = 0$, $T(\{0\}) = \{T(0)\} = \{0\}$. So $\{0\}$ is T-invariant.

(b) Since T maps V to V, for each $v \in V$ we have $T(v) \in V$. Thus $T(V) = \{T(v) \mid v \in V\} \subseteq V$. So V is T-invariant.

(c) Let $v \in \mathsf{R}(T)$. Then $v \in V$. So $T(v) \in T(V) = \mathsf{R}(T)$. As $v \in \mathsf{R}(T)$ is arbitrary, $T(\mathsf{R}(T)) \subseteq \mathsf{R}(T)$. So $\mathsf{R}(T)$ is T-invariant.

(d) Let $v \in \mathsf{N}(T)$. Then $T(v) = 0$. As $T(0) = 0$, $T(v) = 0 \in \mathsf{N}(T)$. As $v \in \mathsf{N}(T)$ is arbitrary, $T(\mathsf{N}(T)) \subseteq \mathsf{N}(T)$ and so $\mathsf{N}(T)$ is T-invariant.

Q2.1.35.

Solution:

(a) Since $V = \mathsf{R}(T) + \mathsf{N}(T)$, to show that $V = \mathsf{R}(T) \oplus \mathsf{N}(T)$ it suffices to show that $\mathsf{R}(T) \cap \mathsf{N}(T) = \{0\}$.

Since V is finite dimensional, $\mathsf{R}(T)$, $\mathsf{N}(T)$, $\mathsf{R}(T) \cap \mathsf{N}(T)$ are all finite dimensional. By dimension theorem, $\dim V = \operatorname{nullity} T + \operatorname{rank} T$. So by Q1.6.29(a), $\dim(\mathsf{R}(T) \cap \mathsf{N}(T)) = \dim \mathsf{R}(T) + \dim \mathsf{N}(T) - \dim(\mathsf{R}(T) + \mathsf{N}(T)) = \operatorname{rank} T + \operatorname{nullity} T - \dim V = 0$. This implies that $\mathsf{R}(T) \cap \mathsf{N}(T) = \{0\}$.

(b) Since $\mathsf{R}(T) \cap \mathsf{N}(T) = \{0\}$, to show that $V = \mathsf{R}(T) \oplus \mathsf{N}(T)$ it suffices to show that $\mathsf{R}(T) + \mathsf{N}(T) = V$.

Since V is finite dimensional, $\mathsf{R}(T)$, $\mathsf{N}(T)$, $\mathsf{R}(T) \cap \mathsf{N}(T)$ are all finite dimensional. By dimension theorem, $\dim V = \operatorname{nullity} T + \operatorname{rank} T$. So by Q1.6.29(a), $\dim(\mathsf{R}(T) + \mathsf{N}(T)) = \dim \mathsf{R}(T) + \dim \mathsf{N}(T) - \dim(\mathsf{R}(T) \cap \mathsf{N}(T)) = \operatorname{rank} T + \operatorname{nullity} T - \dim\{0\} = \operatorname{rank} T + \operatorname{nullity} T = \dim V$. As $\mathsf{R}(T) + \mathsf{N}(T) \subseteq V$, this implies that $\mathsf{R}(T) + \mathsf{N}(T) = V$.

Note

Note that the requirement that V is finite dimensional is necessary: please refer to Q2.1.36.

You can also show the propositions by passing the bases between the spaces.

Optional Part

Q2.1.1.

Solution:

(a) True

(b) False. Consider the conjugation map defined as a map from $\mathbb{C}$ to $\mathbb{C}$ with $\mathbb{C}$ as a complex vector space. Note that the proposition may still hold in certain situations (e.g. Q2.1.37)

(c) False. This only holds if T is linear.

(d) True

(e) False. This only holds if $\dim V = \dim W$.

(f) False. Consider the case where T maps every vector to 0_W.

(g) True

(h) False. Consider the case where $x_1 = 0_V$ but $y_1 \neq 0_W$.

Q2.1.2.

Solution: Let $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$, $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} T(x+y) = T(x+y) &= T(x_1+y_1, x_2+y_2, x_3+y_3) \\ &= (x_1+y_1-x_2-y_2, 2(x_3+y_3)) \\ &= (x_1-x_2, 2x_3) + (y_1-y_2, 2y_3) \\ &= T(x) + T(y) \end{aligned}$$

and

$$\begin{aligned} T(\alpha x) &= T(\alpha(x_1, x_2, x_3)) = T(\alpha x_1, \alpha x_2, \alpha x_3) \\ &= (\alpha x_1 - \alpha x_2, 2\alpha x_3) \\ &= \alpha(x_1 - x_2, 2x_3) \\ &= \alpha T(x) \end{aligned}$$

As x, y, α are arbitrary, T is linear.

The kernel of T is

$$\begin{aligned} \mathsf{N}(T) &= \left\{ x \in \mathbb{R}^3 \;\middle|\; T(x) = 0 \right\} \\ &= \{ (x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}, (x_1 - x_2, 2x_3) = (0, 0) \} \\ &= \{ (x, x, 0) \mid x \in \mathbb{R} \} \\ &= \operatorname{Span}(\{(1, 1, 0)\}) \end{aligned}$$

It is easy to see that $\{(1,1,0)\}$ is a basis of $\mathsf{N}(T)$, so $\operatorname{nullity} T = \dim \mathsf{N}(T) = 1$. Similarly the range of T is

$$\begin{aligned} \mathsf{R}(T) &= \left\{ T(x) \;\middle|\; x \in \mathbb{R}^3 \right\} \\ &= \left\{ (x_1 - x_2, 2x_3) \;\middle|\; (x_1, x_2, x_3) \in \mathbb{R}^3 \right\} \\ &= \mathbb{R}^2 \end{aligned}$$

So $\operatorname{rank} T = \dim \mathsf{R}(T) = 2$.

Since $\dim \mathbb{R}^3 = 3$, we have $\dim \mathbb{R}^3 = \operatorname{nullity} T + \operatorname{rank} T$. So the dimension theorem is verified for T.

Since $\operatorname{nullity} T = 1 \neq 0$, T is not one-to-one. Since $\mathsf{R}(T) = \mathbb{R}^2$, T is onto.

Q2.1.7.

Solution:

(a) $T(0_V) = T(0 \cdot 0_V) = 0 \cdot T(0_V) = 0_W$

(b) Suppose T is linear. Then $T(cx+y) = T(cx) + T(y) = cT(x) + T(y)$ for all $x, y \in V$ and $c \in F$.

Suppose $T(cx+y) = cT(x) + T(y)$ for all $x, y \in V$ and $c \in F$. Take $c = 1$ we have $T(x+y) = T(x) + T(y)$ for all $x, y \in V$. Take $y = 0_V$ we have $T(cx) = cT(x)$ for all $x \in V$, $c \in F$. So T is linear.

(c) $T(x-y) = T(x + (-1) \cdot y) = T(x) + T((-1) \cdot y) = T(x) + (-1) \cdot T(y) = T(x) - T(y)$ for all $x, y \in V$.

(d) Suppose T is linear. Then $T(\sum_{i=1}^n a_i x_i) = \sum_{i=1}^n T(a_i x_i) = \sum_{i=1}^n a_i T(x_i)$ for all $x_1, \ldots, x_n \in V$, $a_1, \ldots, a_n \in F$.

Assume $n \geq 2$. Suppose $T(\sum_{i=1}^n a_i x_i) = \sum_{i=1}^n a_i T(x_i)$ for all $x_1, \ldots, x_n \in V$, $a_1, \ldots, a_n \in F$. Take $x_3 = \ldots = x_n = 0_v$ we have $T(a_1 x_1 + a_2 x_2) = T(\sum_{i=1}^n a_i x_i) = \sum_{i=1}^n a_i T(x_i) = a_1 T(x_1) + a_2 T(x_2)$ for all $x_1, x_2 \in V$, $a_1, a_2 \in F$. Further take $x_2 = 0_v$, we have $T(a_1 x_1) = a_1 T(x_1)$ for all $x_1 \in V$, $a_2 \in F$. On the other hand, take $a_1 = a_2 = 1$ we have $T(x_1 + x_2) = T(x_1) + T(x_2)$ for all $x_1, x_2 \in V$. So T is linear.

Q2.1.9.

Solution:

(a) $T(0_{\mathbb{R}^2}) = T(0,0) = (1,0) \neq 0_{\mathbb{R}^2}$. So T is not linear.

(b) $T(2 \cdot (1,1)) = T(2,2) = (2,4) \neq (2,2) = 2 \cdot T(1,1)$. So T is not linear.

(c) $T(2 \cdot (\frac{\pi}{2}, 0)) = T(\pi, 0) = (0,0) \neq (2,0) = 2 \cdot T(\frac{\pi}{2}, 0)$. So T is not linear.

(d) $T(-2 \cdot (-1,0)) = T(2,0) = (2,0) \neq (-2,0) = -2 \cdot T(-1,0)$. So T is not linear.

(e) $T(0_{\mathbb{R}^2}) = T(0,0) = (0+1,0) = (1,0) \neq (0,0) = 0_{\mathbb{R}^2}$. So T is not linear.

Q2.1.12.

Solution: Suppose there exists such linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$. Then $(2,1) = T(-2,0,-6) = T(2 \cdot (1,0,3)) = 2 \cdot T(1,0,3) = 2 \cdot (1,1) = (2,2)$. Contradiction arises. So no such linear map exists.

Q2.1.15.

Solution: Let $f(x), g(x) \in P(\mathbb{R})$, $\alpha \in \mathbb{R}$. Then by the property of integration we have $T(f(x) + g(x)) = \int_0^x (f + g)(t)\mathrm{d}t = \int_0^x f(t)\mathrm{d}t + \int_0^x g(t)\mathrm{d}t = T(f(x)) + T(g(x))$, $T(\alpha f(x)) = \int_0^x \alpha f(t)\mathrm{d}t = \alpha \int_0^x f(t)\mathrm{d}t = \alpha T(f(x))$. As f, g, α are arbitrary, T is a linear transformation.

Let $f(x) = \sum_{i=0}^n a_i x^i \in P(\mathbb{R})$ with $a_0, \ldots, a_n \in \mathbb{R}$ such that $0 = T(f(x)) = \int_0^x f(t)\mathrm{d}t = \int_0^x \sum_{i=0}^n a_i t^i \mathrm{d}t = \sum_{i=0}^n a_i \int_0^x t^i \mathrm{d}t = \sum_{i=0}^n \frac{a_i}{i+1} x^{i+1}$. Since $\{1, x, \ldots, x^{n+1}\}$ forms a basis, for $i \in \{0, \ldots, n\}$ we have $\frac{a_i}{i+1} = 0$, so $a_i = 0$. Thus $f(x) = 0$ is the zero polynomial. Since the zero polynomial is the only polynomial in the kernel, T is one-to-one.

By the computation above, a polynomial of degree $n \geq 0$ is mapped by T to a polynomial of degree $n+1$, and $T(0) = 0$. Since a nonzero polynomial must have its image degree $n+1 \geq 1$, no polynomial is mapped to polynomial of degree 0. In particular, the constant polynomial $g(x) = 1$ is not in the range. So T is not onto.

Q2.1.16.

Solution: Let $f(x) = \sum_{i=0}^n a_i x^i \in P(\mathbb{R})$ with $a_0, \ldots, a_n \in \mathbb{R}$. Let $g(x) = \sum_{i=0}^n \frac{a_i}{i+1} x^{i+1}$. Then $g(x) \in P(\mathbb{R})$, and $T(g(x)) = g'(x) = \sum_{i=0}^n \frac{a_i}{i+1} \frac{\mathrm{d}}{\mathrm{d}x} x^{i+1} = \sum_{i=0}^n \frac{a_i}{i+1}(i+1)x^i = \sum_{i=0}^n a_i x^i = f(x)$. So $f(x) \in \mathsf{R}(T)$. As $f(x) \in P(\mathbb{R})$ is arbitrary, T is onto.

Let $h(x) = 1$ be the constant 1 function. Then $h(x) \in P(\mathbb{R}) \setminus \{0\}$, and $T(h(x)) = \frac{\mathrm{d}}{\mathrm{d}x} 1 = 0 = T(0)$. So T is not one-to-one.

Q2.1.18.

Solution: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x,y) = (y,0)$ for $(x,y) \in \mathbb{R}^2$. It is easy to see that T is linear.

The kernel of T is $\mathsf{N}(T) = \{\, (x,y) \mid x, y \in \mathbb{R}, (y,0) = (0,0) \,\} = \{\, (x,0) \mid x \in \mathbb{R} \,\} = \mathrm{Span}\,(\{(1,0)\})$. The range of T is $\mathsf{R}(T) = \left\{\, T(x,y) \;\middle|\; (x,y) \in \mathbb{R}^2 \,\right\} = \{\, (y,0) \mid y \in \mathbb{R} \,\} = \mathrm{Span}\,(\{(1,0)\})$. So $\mathsf{N}(T) = \mathsf{R}(T)$.

Q2.1.19.

Solution: Let $V = \mathbb{R}$ be the real line as a real vector space. Let $T, U : \mathbb{R} \to \mathbb{R}$ be defined as $T(x) = x$, $U(x) = -x$ for $x \in V$. It is easy to see that T, U are linear.

Also, it is easy to verify that $\mathsf{N}(T) = \{0\} = \mathsf{N}(U)$, and $\mathsf{R}(T) = V = \mathsf{R}(U)$.

Q2.1.21.

Solution:

(a) Let $v = (v_1, v_2, \ldots), w = (w_1, w_2, \ldots) \in V$, $\alpha \in F$. Then $T(v+w) = T((v_1+w_1, v_2+w_2, \ldots)) = (v_2+w_2, v_3+w_3, \ldots) = (v_2, v_3, \ldots) + (w_2, w_3, \ldots) = T(v) + T(w)$, $U(v+w) = U((v_1+w_1, v_2+w_2, \ldots)) = (0, v_1+w_1, v_2+w_2, \ldots) = (0, v_1, v_2, \ldots) + (0, w_1, w_2, \ldots) = U(v) + U(w)$, $T(\alpha v) = T((\alpha v_1, \alpha v_2, \ldots)) = (\alpha v_2, \alpha v_3, \ldots) = \alpha(v_2, v_3, \ldots) = \alpha T(v)$, $U(\alpha v) = U((\alpha v_1, \alpha v_2, \ldots)) = (0, \alpha v_1, \alpha v_2, \ldots) = \alpha(0, v_1, v_2, \ldots) = \alpha U(v)$. Since v, w, α are arbitrary, T, U are linear.

(b) Let $v = (v_1, v_2, \ldots) \in V$. Then for $w = (0, v_1, v_2, \ldots)$, $w \in V$, $T(w) = (v_1, v_2, \ldots) = v \in \mathsf{R}(T)$. Since $v \in V$ is arbitrary, T is onto.

Let $x = (1, 0, 0, \ldots) \in V$ be the sequence such that only the first entry is 1 and all other entries are 0. Then $T(x) = (0, 0, \ldots) = \vec{0}$ is the zero sequence. As $x \neq \vec{0}$ but $T(x) = \vec{0}$, T is not one-to-one.

(c) Let $v = (v_1, v_2, \ldots) \in V$ such that $U(v) = \vec{0}$. Then $(0, 0, 0, \ldots) = U((v_1, v_2, \ldots)) = (0, v_1, v_2, \ldots)$, so $v_1 = v_2 = \ldots = 0$. This implies that $v = (v_1, v_2, \ldots) = (0, 0, \ldots) = 0$. As U is linear, U is one-to-one.

Let $w = (1, 0, 0, \ldots) \in V$ be the sequence where only the first entry is 1 and all other entries are 0. Then for all $v = (v_1, v_2, \ldots) \in V$, $U(v) = (0, v_1, v_2, \ldots) \neq (1, 0, 0, \ldots)$. So $w \notin \mathsf{R}(U)$. This implies that U is not onto.

Q2.1.26.

Solution:

(a) Let $v, u \in V$, $\alpha \in F$. As $V = W_1 \oplus W_2$, there are unique decompositions $v = v_1 + v_2$, $u = u_1 + u_2$ where $v_1, u_1 \in W_1$, $v_2, u_2 \in W_2$. We also have $v + u = (v_1 + u_1) + (v_2 + u_2)$ and $\alpha v = \alpha v_1 + \alpha v_2$ with $v_1 + u_1, \alpha v_1 \in W_1, v_2 + u_2, \alpha v_2 \in W_2$. Then by definition, $T(v + u) = T((v_1 + u_1) + (v_2 + u_2)) = v_1 + u_1 = T(v) + T(u)$, $T(\alpha v) = T(\alpha v_1 + \alpha v_2) = \alpha v_1 = \alpha T(v)$. Since v, u, α are arbitrary, T is linear.

Denote $S = \{x \in V \mid T(x) = x\}$.

Let $x \in W_1$. Then the unique decomposition of x is $x = x + 0$ where $x \in W_1$ and $0 \in W_2$. Then by definition, $T(x) = x$, so $x \in S$. As $x \in W_1$ is arbitrary, $W_1 \subseteq S$.

Let $x \in S \subseteq V$. Since $V = W_1 \oplus W_2$, there exists unique $x_1 \in W_1, x_2 \in W_2$ such that $x = x_1 + x_2$. By definition, $x = T(x) = x_1 \in W_1$. As $x \in S$ is arbitrary, $S \subseteq W_1$.

Therefore $W = S = \{x \in V \mid T(x) = x\}$.

(b) By the definition of projection, $\mathsf{R}(T) \subseteq W_1$. By the previous part, $W \subseteq \mathsf{R}(T)$. So $W = \mathsf{R}(T)$.

Let $v \in W_2$. Then the decomposition of v is $v = 0 + v$ where $0 \in W_1, v \in W_2$. So $T(v) = 0$, $v \in \mathsf{N}(T)$. As v is arbitrary, $W_2 \subseteq \mathsf{N}(T)$.

Let $v \in \mathsf{N}(T)$. Let the decomposition of v is $v = v_1 + v_2$ where $v_1 \in W_1, v_2 \in W_2$. Then $0 = T(v) = v_1$, so $v = v_1 + v_2 = v_2 \in W_2$. As v is arbitrary, $\mathsf{N}(T) \subseteq W_2$. Therefore $W_2 = \mathsf{N}(T)$.

(c) The only possible choice for W_2 with $V = W_1 \oplus W_2$ is $W_2 = \{0\}$. As every vector $v \in V$ can be presented as $v = v + 0$ with $v \in V$, $T(v) = v$ for all $v \in V$. So T is the identity map.

(d) The only possible choice for W_2 with $V = W_1 \oplus W_2$ is $W_2 = V$. Also, $\mathsf{R}(T) = W_1 = \{0\}$, T is the zero map.

Q2.1.27.

Solution:

(a) By the last part of the previous question, when $W = V$, $W' = \{0\}$ and T is the identity map. Similarly, when $W = \{0\}$, $W' = V$ and T is the zero map. So we may assume that W is a nontrivial proper subspace of V.

Since V is finite dimensional, W is also finite dimensional. Let $\{w_1, \ldots, w_n\}$ be a basis of W. By Extension Theorem we may extend $\{w_1, \ldots, w_n\}$ to a basis $\{w_1, \ldots, w_n, v_1, \ldots, v_m\}$ of V by appending $v_1, \ldots, v_m \in V$. Let $W' = \operatorname{Span}(\{v_1, \ldots, v_m\})$. By the definition of basis, it is easy to verify that $V = W \oplus W'$.

Since $\{w_1, \ldots, w_n, v_1, \ldots, v_m\}$ is a basis of V, by Theorem 1.8 for each $v \in V$ there exists unique scalars $a_1, \ldots, a_n, b_1, \ldots, b_m$ such that $v = \sum_{i=1}^n a_i w_i + \sum_{i=1}^m b_i v_i$. We define $T : V \to V$ be such that $T(v) = \sum_{i=1}^n a_i w_i$. By the uniqueness of a_i, T is well-defined.

We now show that T is a linear map and is a projection on W along W'.

Let $v, v' \in V$, $\alpha \in F$. Then there exists unique $a_1, \ldots, a_n, b_1, \ldots, b_m, a'_1, \ldots, a'_n, b'_1, \ldots, b'_m \in F$ such that $v = \sum_{i=1}^n a_i w_i + \sum_{i=1}^m b_i v_i$, $v' = \sum_{i=1}^n a'_i w_i + \sum_{i=1}^m b'_i v_i$. So $T(v) = \sum_{i=1}^n a_i w_i$, $T(v') = \sum_{i=1}^n a'_i w_i$. By uniqueness of the coefficients, we have $v + v' = \sum_{i=1}^n (a_i + a'_i) w_i + \sum_{i=1}^m (b_i + b'_i) v_i$ and $\alpha v = \sum_{i=1}^n \alpha a_i w_i + \sum_{i=1}^m \alpha b_i v_i$, so $T(v + v') = \sum_{i=1}^n (a_i + a'_i) w_i = T(v) + T(v')$ and $T(\alpha v) = \sum_{i=1}^n \alpha a_i w_i = \alpha T(v)$. As v, v', α are arbitrary, T is linear.

Let $x = x_1 + x_2 \in V$ with $x_1 \in W$, $x_2 \in W'$. By the property of bases, there exists unique $a_1, \ldots, a_n, b_1, \ldots, b_n \in F$ such that $x_1 = \sum_{i=1}^n a_i w_i$ and $x_2 = \sum_{i=1}^m b_i v_i$. So $x = x_1 + x_2 = \sum_{i=1}^n a_i w_i + \sum_{i=1}^m b_i v_i$. By the uniqueness of the coefficients and the definition of T we have $T(x) = \sum_{i=1}^n a_i w_i = x_1$. Since x_1, x_2 are arbitrary, T is a projection on W along W'.

(b) Consider $V = \mathbb{R}^2$ be the usual real plane, $W = \{(x, 0) \mid x \in \mathbb{R}\}$ be the x-axis. Let $W_1 = \{(0, y) \mid y \in \mathbb{R}\}$, $W_2 = \{(x, x) \mid x \in \mathbb{R}\}$. It is easy to verify that W, W_1, W_2 are subspaces of V, and $V = W \oplus W_1 = W \oplus W_2$. The corresponding projection maps are respectively $T_1(x, y) = (x, 0)$ and $T_2(x, y) = (x - y, 0)$ for $(x, y) \in V$.

Note

Please refer to Q1.3.31, Q2.1.26, Q2.1.40 and this note by Prof. B. Binegar from OSU for a better understanding.

Q2.1.31.

Solution:

(a) Let $w \in W$, $v = T(w) \in \mathsf{R}(T)$. Since $V = \mathsf{R}(T) \oplus W$, $\mathsf{R}(T) \cap W = \{0\}$. As W is T-invariant, $v = T(w) \in W$. So $v \in \mathsf{R}(T) \cap W = \{0\}$, so $v = 0$, $w \in \mathsf{N}(T)$. As $w \in W$ is arbitrary, $W \subseteq \mathsf{N}(T)$.

(b) Since V is finite dimensional, we may use the dimension theorem.

By the dimensional theorem, $\dim V = \mathsf{N}(T) + \operatorname{rank} T$. By the previous part, $\dim W \leq \dim \mathsf{N}(T) = \operatorname{nullity} T$. Since $V = \mathsf{R}(T) + W$, we have $\dim V \leq \dim \mathsf{R}(T) + \dim W = \operatorname{rank} T + \dim W \leq \operatorname{rank} T + \operatorname{nullity} T = \dim V$. So all equal signs must hold. In particular, $\dim W = \operatorname{nullity} T = \dim \mathsf{N}(T)$. Since $W \subseteq \mathsf{N}(T)$, $W = \mathsf{N}(T)$.

(c) Consider V be the space of all real sequences equipped with entry-wise addition and scalar multiplication, $T : V \to V$ is the right shift operator, and $W = \{0\}$ is the trivial subspace. Then $\mathsf{R}(T) = V$, but $W = \{0\} \neq \{(0, a_2, \ldots) \mid a_2, \ldots \in \mathbb{R}\} = \mathsf{N}(T)$.

Note

Please refer to the dimension theorem (for infinite dimensional spaces).

Q2.1.32.

Solution: Let $v \in \mathsf{N}(T_W)$. Then $T_W(v) = 0$. In particular, T_W is defined for $v \in V$, so $v \in W$. Also, $0 = T_W(v) = T(v)$, so $v \in \mathsf{N}(T)$. Thus $v \in W \cap \mathsf{N}(T)$. As $v \in \mathsf{N}(T_W)$ is arbitrary, $\mathsf{N}(T_W) \subseteq W \cap \mathsf{N}(T)$.

Let $v \in \mathsf{N}(T) \cap W$. Then T_W is defined for v. So $T_W(v) = T(v) = 0$. Hence $v \in \mathsf{N}(T_W)$. As v is arbitrary, $\mathsf{N}(T) \cap W \subseteq \mathsf{N}(T_W)$.

So $\mathsf{N}(T_W) = \mathsf{N}(T) \cap W$.

Q2.1.37.

Solution: To show that T is linear, it suffices to show that $T(\lambda v) = \lambda T(v)$ for all $\lambda \in \mathbb{Q}$, $v \in V$.

By additivity, $T(0_V) = T(0_V + 0_V) = T(0_V) + T(0_V)$, so $T(0_V) = 0_W$.

Also, for each $v \in V$, $0_W = T(0_V) = T(v + (-v)) = T(v) + T(-v)$, so $T(-v) = -T(v)$.

Trivially $T(1 \cdot v) = T(v) = 1 \cdot T(v)$ for all $v \in V$.

Suppose for some $n \in \mathbb{Z}^+$, $T(n \cdot v) = n \cdot T(v)$ for all $v \in V$. Then for $v \in V$, $T((n+1) \cdot v) = T(n \cdot v + v) = T(n \cdot v) + T(v) = n \cdot T(v) + T(v) = (n+1) \cdot T(v)$ as T is additive.

So by induction, $T(n \cdot v) = n \cdot T(v)$ for all $v \in V$, $n \in \mathbb{Z}$.

For $n \in \mathbb{Z} \setminus \{0\}$ and $v \in V$, $T(v) = T(n \cdot (\frac{1}{n} v)) = n \cdot T(\frac{1}{n} v)$, so $T(\frac{1}{n} v) = \frac{1}{n} T(v)$.

Let $v \in V$, $\lambda = \frac{p}{q} \in \mathbb{Q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$. Then $T(\lambda v) = T(\frac{p}{q} v) = T(p \cdot \frac{1}{q} \cdot v) = pT(\frac{1}{q} \cdot v) = p \cdot \frac{1}{q} \cdot T(v) = \frac{p}{q} T(v) = \lambda T(v)$. As v, λ are arbitrary, T is linear.

Note

Note that this requires the scalar field of the vectors to be small enough (so that the every scalar is simply a ratio of the integers).