MATH2040A Homework 1 suggested answer

Compulsory Part

Q1.2.8.

Solution: Let $a, b \in F$, $x, y \in V$.

By the axioms of vector space, $(a+b)(x+y) = (a+b)x + (a+b)y = (ax+bx) + (ay+by) = ax+ay+bx+by$.

Q1.2.13.

Solution: V is not a vector space over $\mathbb{R}$.

There are multiple ways to show this. One way is as follows:

Suppose V is a vector space over $\mathbb{R}$. Then by the axioms of vector space, $(0,2) = 2 \cdot (0,2) = (1+1) \cdot (0,2) = 1 \cdot (0,2) + 1 \cdot (0,2) = (0,2) + (0,2) = (0,4)$. Contradiction arises.

You can also prove this by e.g. checking the properties of the supposed zero vector, or checking the invertibility of a certain vector.

Note

To show that it is not a vector space, you only need to show that some axioms do not hold. You can of course verify all other axioms, but that would be unnecessary.

Also note that your counterexample should be constructed clearly. If your counterexample contains some free variables, you should argue that those variables can be bounded arbitrary, i.e. every possible choice of values gives a counterexample.

Also note that Theorem 1.3 in the textbook cannot be applied here. The theorem can only be applied if you already have a vector space and want to investigate its subset.

Also note that the (purposed) zero vector is not $(0,0)$. Although the base set $\mathbb{R}^2$ also has a vector space structure with the tuple $(0,0)$ being its zero vector, *it does not mean that every vector space built upon it must have the same zero vector*. For example, consider the set $\mathbb{R}^2$ equipped the following operations: $(a,b)+(c,d) = (a+b-1, c+d-1)$, $\lambda(a,b) = (1+\lambda(a-1), 1+\lambda(b-1))$. It is easy to see that such set is also a vector space with $(1,1)$ being its zero vector. (In fact, it is just shifting the usual $\mathbb{R}^2$ by $(1,1)$.)

Q1.2.21.

Solution: Z is indeed a vector space with the operations defined. To prove this, the simplest way is to verify all axioms one by one. For brevity, we will only give the proofs of VS 3 and VS 4 here. Others are straightforward.

(VS 3) Since V, W are vector spaces, they have their zero vectors $0_V \in V$, $0_W \in W$. Let $\vec{0} = (0_V, 0_W) \in Z$. Then $\vec{0}$ is the zero vector of Z: for arbitrary $(v,w) \in Z$, $(v,w) + \vec{0} = (v,w) + (0_V, 0_W) = (v + 0_V, w + 0_W) = (v,w)$.

(VS 4) Let $\vec{x} = (v,w) \in Z$ be arbitrary with $v \in V$, $w \in W$.

Since V, W are vector spaces, v and w have their additive inverses $v' \in V$, $w' \in W$ such that $v + v' = 0_V$, $w + w' = 0_W$.

Let $\vec{y} = (v', w') \in Z$. Then $\vec{x} + \vec{y} = (v, w) + (v', w') = (v + v', w + w') = (0_V, 0_W) = \vec{0}$, which is the zero vector of Z. So $\vec{y}$ is an additive inverse of $\vec{x}$.

As $\vec{x} \in Z$ is chosen arbitrarily, every element in Z has an additive inverse.

Note

Note that the zero vector in VS 3 should be constructed clearly. You cannot simply put a "(0, 0)" or just "0" as your zero vector without stating what how it is defined. Similar for additive inverse in VS 4.

Also, you have to verify all 8 axioms. Theorem 1.3 in the textbook cannot be applied here, unless you have shown prior that Z is a subset of some vector space.

Also, you cannot simply assume that such zero vector exists, then proceed to find such vector without further verifying its property. In particular, the following argument is considered incomplete:

> Assume that a zero vector exists.
>
> > Let this vector be v.
> >
> > ......
> >
> > So we can solve some equation to see that $v = \textit{something}$.
>
> Therefore there exists a zero vector.

Such argument does not constitute as a valid proof as it only shows this proposition:

> "v exists" $\Rightarrow$ "$v = \textit{something}$"

However, it has never shown the existence of such v (i.e. the antecedent of the above proposition). To complete the proof, you should

1. Show that the v you found can be constructed without assuming it is a zero vector (unless it is trivial); then
2. Show that the v you constructed is indeed a zero vector

or simply start the proof from step 2 if you already know what the zero vector is.

Q1.3.11.

Solution: Consider $f(x) = x^n + 1$ and $g(x) = -x^n$.

By definition, $f(x), g(x) \in W$ as $\deg f = \deg g = n$. However, $f(x) + g(x) = (x^n + 1) - x^n = 1$, which is not in W as $f(x) + g(x) \neq 0$ and $\deg(f(x) + g(x)) = \deg 1 = 0$.

Since W is not closed under addition, W is not a subspace of $\mathrm{P}(F)$.

Note

It is acceptable if your proof is only for certain n (e.g. only for $n = 2$), but in principle your proof should cover all case for $n \geq 1$.

Also note that $\deg 0$ is not universally defined. Depending on the context, $\deg 0$ can be $-\infty$, -1, or simply left undefined.

Q1.3.19.

Solution: Suppose $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. Without loss of generality, we assume $W_1 \subseteq W_2$. Then $W_1 \cup W_2 = W_2$. Since W_2 is already a subspace, $W_1 \cup W_2$ is a subspace.

Suppose on the other hand neither $W_1 \subseteq W_2$ nor $W_2 \subseteq W_1$. Then there exists $w_1 \in W_1 \setminus W_2$ and $w_2 \in W_2 \setminus w_1$. By definition, $w_1, w_2 \in W_1 \cup W_2$. Let $x = w_1 + w_2 \in V$.

- If $x \in W_1$, then $x - w_1 = (w_1 + w_2) - w_1 = w_2 \in W_1$ (as W_1 is a subspace). Contradiction arises as $w_2 \notin W_1$.
- If $x \in W_2$, then $x - w_2 = (w_1 + w_2) - w_2 = w_1 \in W_2$ (as W_2 is a subspace). Contradiction arises as $w_1 \notin W_2$.

Therefore $x \notin W_1$ and $x \notin W_2$. In particular $x \notin W_1 \cup W_2$. As $W_1 \cup W_2$ is not closed under addition, it is not a subspace.

Note

Please refer to Optional Part Q1.3.23 for a better understanding of $W_1 \cup W_2$.

Q1.3.22.

Solution: Denote the set of even functions in $\mathcal{F}(F_1, F_2)$ by $\mathcal{E}$, and the set of odd functions in $\mathcal{F}(F_1, F_2)$ by $\mathcal{O}$.

We prove the proposition using Theorem 1.3 in textbook. To do so, we need to show that

1. The zero vector of $\mathcal{F}(F_1, F_2)$ is in both $\mathcal{E}$ and $\mathcal{O}$. In particular both are nonempty;
2. Both sets are closed under addition;
3. Both sets are closed under scalar multiplication.

Let $\vec{0}(x)$ be the zero function in $\mathcal{F}(F_1, F_2)$ be defined by $\vec{0}(x) = 0_{F_2}$ for all $x \in F_1$, where 0_{F_2} is the zero element of F_2. Trivially $\vec{0}$ is the zero vector of $\mathcal{F}(F_1, F_2)$, and $\vec{0}(x)$ is in both $\mathcal{E}$ and $\mathcal{O}$.

Let $f(x), g(x) \in \mathcal{E}$. Then for all $x \in F_1$, $f(-x) = f(x)$, $g(-x) = g(x)$. Then for all $x \in F_1$, $(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$. So $f + g \in \mathcal{E}$. Since $f, g \in \mathcal{E}$ are arbitrary, $\mathcal{E}$ is closed under addition.

Let $f(x) \in \mathcal{E}$, $\lambda \in F_2$. Then for all $x \in F_1$, $(\lambda f)(-x) = \lambda f(-x) = \lambda f(x) = (\lambda f)(x)$, so $\lambda f \in \mathcal{E}$. Since f, λ are arbitrary, $\mathcal{E}$ is closed under scalar multiplication.

By Theorem 1.3, $\mathcal{E}$ is a subspace of $\mathcal{F}(F_1, F_2)$.

The proof for $\mathcal{O}$ is similar and so is omitted here for brevity.

Note

Note that $\mathcal{F}(F_1, F_2)$ is a vector space over the scalar field F_2.

Optional Part

Q1.2.1.

Solution:

(a) True

(b) False. Consider $\{\vec{0}\}$, the trivial vector space that contains only the zero vector

(c) False. Consider $x = \vec{0}$

(d) False. Consider $a = 0$

(e) True

(f) False

(g) False

(h) False. Refer to Question 1.3.11

(i) True

(j) True

(k) True

Q1.2.14.

Solution: V is a vector over $\mathbb{R}$. The proof is a simple verification of all 8 axioms and is omitted here6.

Note

Theorem 1.3 cannot be applied here as $(V, \mathbb{R})$ is not a subspace of $(V, \mathbb{C})$: they have different scalar fields.

Q1.2.15.

Solution: V is not a vector space as it is not closed under scalar multiplication: consider the vector $(1, 0, \ldots, 0)$ and the scalar i. Easy to see that $i \cdot (1, 0, \ldots, 0)$ is not a real vector.

Q1.2.20.

Solution: The proof is a simple verification of all 8 axioms. The zero vector is the zero sequence $\vec{0} = \{0\}$ where every entry is 0. The additive inverse of the sequence $\{a_n\}$ is $\{-a_n\}$, where every entry is the additive inverse of the corresponding entry.

Q1.3.1.

Solution:

(a) False. Consider $V = \mathbb{C}$ as a complex vector space and $W = \mathbb{C}$ as a real vector space.

(b) False. Note the difference between a *subset* and a *subspace*

(c) True

(d) False. Consider the intersection being empty

(e) True

(f) False

(g) False. They are only *isomorphic*, meaning they can be identified as having the same structure, but they are still *not equal*

Q1.3.8.

Solution:

(a) Yes

(b) No. $\vec{0} \notin W_2$

(c) Yes

(d) Yes

(e) No. $\vec{0} \notin W_5$

(f) No. $(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{3}}, 0), (0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}) \in W_6$, but their sum is not

The proof for the subsets being subspaces are simple applications of related theorems (e.g. Theorem 1.3) and so are omitted here.

Q1.3.17.

Solution: Suppose W is a subspace of V. Then by definition of subspace, $\vec{0} \in W$, so $W \neq \emptyset$. The other two conditions also follow from the definition of subspace.

Suppose $W \neq \emptyset$ and $ax, x+y \in W$ whenever $a \in F$, $x, y \in W$. Since $W \neq \emptyset$, there exists a vector $z \in W$. Then by assumption, $-z = (-1) \cdot z \in W$, and so $\vec{0} = z + (-z) \in W$. By theorem 1.3, W is a subspace.

Q1.3.23.

Solution: Recall by definition $W_1 + W_2 = \{ w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2 \}$.

(a) We need to show two things: $W_1 + W_2$ is a subspace, and $W_1 + W_2$ contains both W_1 and W_2.

To show that $W_1 + W_2$ is a subspace, we use Theorem 1.3:

- Since W_1, W_2 are subspaces, we have $\vec{0} \in W_1, W_2$. So $\vec{0} = \vec{0} + \vec{0} \in W_1 + W_2$.
- Let $x, y \in W_1 + W_2$. By definition, there exists $x_1, y_1 \in W_1$, $x_2, y_2 \in W_2$ such that $x = x_1 + x_2$, $y = y_1 + y_2$. As W_1, W_2 are subspaces, $x_1 + y_1 \in W_1$, $x_2 + y_2 \in W_2$. So $x + y = (x_1 + x_2) + (y_1 + y_2) = (x_1 + y_1) + (x_2 + y_2) \in W_1 + W_2$.
- Let $x \in W_1 + W_2$, $\lambda \in \mathbb{F}$ is a scalar. Then there exists $w_1 \in W_1$, $w_2 \in W_2$ such that $x = w_1 + w_2$. Since W_1, W_2 are subspaces, $\lambda w_1 \in W_1$, $\lambda w_2 \in W_2$. So $\lambda x = \lambda(w_1 + w_2) = (\lambda w_1) + (\lambda w_2) \in W_1 + W_2$

By Theorem 1.3, $W_1 + W_2$ is a subspace.

For each $w \in W_1$, $w = w + \vec{0} \in W_1 + W_2$ with $\vec{0} \in W_2$. So $W_1 \subseteq W_1 + W_2$. Similarly, $W_2 \subseteq W_1 + W_2$.

Therefore, $W_1 + W_2$ is a subspace that contains both W_1 and W_2.

(b) Let $U \subseteq V$ be a subspace of V that contains both W_1 and W_2.

Let $w \in W_1 + W_2$. Then there exist $w_1 \in W_1$, $w_2 \in W_2$ such that $w = w_1 + w_2$. By assumption, $w_1, w_2 \in U$. Since U is a subspace, $w = w_1 + w_2 \in U$.

As $w \in W_1 + W_2$ is arbitrary, $W_1 + W_2 \subseteq U$.

Since U is arbitrary, every subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

Q1.3.28.

Solution:

(a) Since $W_1 \subseteq M_{n\times n}(F)$ is already contained in a vector space, we can use Theorem 1.3:

1. Trivially the zero matrix, where every entry is 0, is the zero vector of $M_{n\times n}(F)$ and is also a skew-symmetric matrix
2. Let $M_1, M_2 \in W_1$. Then ${M_1}^\intercal = -M_1$, ${M_2}^\intercal = -M_2$. By the properties of matrices, ${M_1 + M_2}^\intercal = {M_1}^\intercal + {M_2}^\intercal = -M_1 - M_2 = -(M_1 + M_2)$, so $M_1 + M_2 \in W_1$.
3. Let $M \in W_1$, $\lambda \in F$. Then $M^\intercal = -M$. By the properties of matrices, $\lambda M^\intercal = \lambda M^\intercal = \lambda(-M) = -(\lambda M)$.

Therefore W_1 is a subspace of $M_{n\times n}(F)$.

(b) Assume that F is not of characteristic 2. This means that $2_F = 1_F + 1_F \neq 0_F$.

To show that $M_{n\times n}(F) = W_1 \oplus W_2$, we need to show that

- $W_1 \cap W_2 = \{0\}$
- $W_1 + W_2 = M_{n\times n}(F)$

They can be verified directly:

- Let $M \in W_1 \cap W_2$. Then $M = M^\intercal = -M$. So $2_F M = M + M = 0$. This implies $M = 0$. So $W_1 \cap W_2 \subseteq \{0\}$.
  On the other hand, $0 \in W_1$ and $0 \in W_2$, so $\{0\} \subseteq W_1 \cap W_2$. Therefore $W_1 \cap W_2 = \{0\}$.
- Trivially $W_1 + W_2 \subseteq M_{n\times n}(F)$.
  Let $M \in M_{n\times n}(F)$. Let $M_1 = \frac{1}{2_F}(M - M^\intercal)$, $M_2 = \frac{1}{2_F}(M + M^\intercal)$. Then it is easy verify that $M_1 \in W_1$, $M_2 \in W_2$, $M = M_1 + M_2$. So $M \in W_1 + W_2$. As M is arbitrary, $M_{n\times n}(F) \subseteq W_1 + W_2$. Together with the above inclusion, we have $M_{n\times n}(F) = W_1 + W_2$.

So by definition $M_{n\times n}(F) = W_1 \oplus W_2$.

Q1.3.30.

Solution:

(a) Suppose $V = W_1 \oplus W_2$. Then $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. In particular for each $v \in V$ there exists $w_1 \in W_1$, $w_2 \in W_2$ such that $v = w_1 + w_2$.

Suppose for some $v \in V$ there are two ways to decompose v: $v = w_1 + w_2 = w_1' + w_2'$ with $w_1, w_1' \in W_1$, $w_2, w_2' \in W_2$. Then $w_1 + w_2 = w_1' + w_2'$ and so $w_1 - w_1' = w_2 - w_2'$. By properties of subspace, $w_1 - w_1' \in W_1$ and $w_2 - w_2' \in W_2$. This implies that $w_1 - w_1' = w_2 - w_2' \in W_1 \cap W_2 = \{0\}$. So $w_1 = w_1'$, $w_2 = w_2'$, and the way to decompose v is unique.

(b) Suppose every $v \in V$ can be uniquely written as $x_1 + x_2$ with $x_1 \in W_1$, $x_2 \in W_2$.

By assumption, $V \subseteq W_1 + W_2$ and so $V = W_1 + W_2$. To show that $V = W_1 \otimes W_2$, it suffices to show that $W_1 \cap W_2 \subseteq \{0\}$.

Let $v \in W_1 \cap W_2 \subseteq V$. Then $v = v + 0$ with $v \in W_1$, $0 \in W_2$ and $v = 0 + v$ with $0 \in W_1$, $v \in W_2$. By assumption, these two decomposition must be identical. This means that $v = 0$. As $v \in W_1 \cap W_2$ is arbitrary, $W_1 \cap W_2 = \{0\}$.

Q1.3.31.

Solution:

(a) Suppose $v + W$ is a subspace. Then $0 \in v + W = \{ v + w \mid w \in W \}$, so $-v \in W$. Since W is a subspace, $v \in W$.

Suppose on the other hand $v \in W$. By the property of subspace, $-v \in W$, so $0 = v + (-v) \in v + W$.

Let $x, y \in v+W$. Then there exists $w, w' \in W$ such that $x = v+w$, $y = v+w'$. By the property of subspace, $v + w + w' \in W$, so $x + y = (v + w) + (v + w') = v + (v + w + w') \in v + W$.

Let $x \in v+W$, $\lambda \in F$ is a scalar. Then there exists $w \in W$ such that $x = v+w$. By the property of subspace $(\lambda - 1)v \in W$, $\lambda w \in W$, so $(\lambda - 1)v + \lambda w \in W$. Thus $\lambda x = \lambda(v + w) = v + ((\lambda - 1)v + \lambda w) \in v + W$.

By Theorem 1.3, $v + W$ is a subspace of V.

(b) Suppose $v_1 + W = v_2 + W$. Then there exists $w \in W$ such that $v_1 + 0 = v_2 + w$, so $v_1 - v_2 = w \in W$.

Suppose on the other hand that $v_1 - v_2 \in W$. Then $v_2 - v_1 \in W$.

Let $v \in v_1 + W$. Then there exists $w \in W$ such that $v = v_1 + w$. Then $v = v_2 + ((v_2 - v_1) + w) \in v_2 + W$. Since $v \in v_1 + W$ is arbitrary, $v_1 + W \subseteq v_2 + W$.

By similar argument we can also show that $v_2 + W \subseteq v_1 + W$. So $v_1 + W = v_2 + W$.

(c) By part (b), it suffices to show that $(v_1 + v_2) - (v_1' + v_2') \in W$ and $av_1 - av_1' \in W$.

Since $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$. By part (b), $v_1 - v_1', v_2 - v_2' \in W$. So $(v_1 + v_2) - (v_1' + v_2') = (v_1 - v_1') + (v_2 - v_2') \in W$, $av_1 - av_1' = a(v_1 - v_1') \in W$.

(d) For simplicity, we denote $x + W = [x]$.

We verify each axiom of vector space one by one:

1. For $[x], [y] \in S$, $[x] + [y] = [x + y] = [y + x] = [y] + [x]$
2. For $[x], [y], [z] \in S$, $([x] + [y]) + [z] = [x + y] + [z] = [x + y + z] = [x] + [y + z] = [x] + ([y] + [z])$
3. The zero vector in S is $[0] = W$. For each $[x] \in S$, $[x] + [0] = [x + 0] = [x]$
4. For $[x] \in S$ we have $[-x] \in S$ and $[x] + [-x] = [x - x] = [0]$
5. For $[x] \in S$, $1 \cdot [x] = [1 \cdot x] = [x]$
6. For $a, b \in F$ and $[x] \in S$, $(ab) \cdot [x] = [(ab) \cdot x] = [a \cdot (b \cdot x)] = a \cdot [b \cdot x] = a \cdot (b \cdot [x])$
7. For $a \in F$ and $[x], [y] \in S$, $a \cdot ([x]+[y]) = a \cdot [x+y] = [a \cdot (x+y)] = [a \cdot x + a \cdot y] = [a \cdot x]+[a \cdot y] = a \cdot [x]+a \cdot [y]$
8. For $a, b \in F$, $[x] \in S$, $(a + b) \cdot [x] = [(a + b) \cdot x] = [a \cdot x + b \cdot x] = [a \cdot x] + [b \cdot x] = a \cdot [x] + b \cdot [x]$

Since every axiom is satisfied, S is a vector space over F with the operations defined.

Note

For those who have studied some abstract algebra, the construction may seem familiar: it is the (almost) same definition of *quotient set/partition* in set theory, *quotient group* in group theory, *quotient space* in topology, etc. They all in fact fall into to same class called *quotient object* in the theory of *category*, and their universal properties are studied there.

Also, there is a analogy of *the first isomorphism theorem* in group theory on vector spaces, which involves the concepts of linear maps covered in later lectures. Unfortunately, this analogy is not covered in the syllabus and is rarely taught in this course.