MATH2040A Tutorial 1 Exercises Suggested Answer

This is the suggested solution for the extra exercises posed on the first tutorial session of MATH2040A on September 16, 2021. The solutions of the optional part of the homework will be put on the course website together with the compulsory part next week. There are many approaches to solve these exercises, and the solution presented here is only one approach out of many others (that could be more elegant and robust). There might be some theorems covered in lectures that can make the proofs more concise, and some parts of the computations are skipped. Readers are encouraged to solve the problems by themselves first before reading the solutions.

Please contact me if you have any question regarding the solutions here, or if you have found a solution you want to share.

Question

1. Let U_1, U_2, W be subspaces of V. Determine if $W \cap (U_1 + U_2) = (W \cap U_1) + (W \cap U_2)$. What if $U_1 \subseteq W$?

> **Solution:** The assertion is incorrect. One counterexample is as follow:
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> Consider $V = \mathbb{R}^2$ to be the real plane, $U_1 = \{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ be the x-axis, $U_2 = \{(0, y) \in \mathbb{R}^2 : y \in \mathbb{R}\}$ be the y-axis, and $W = \{(x, x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$. It is easy to verify that U_1, U_2, W are subspaces of V. Then we have $W \cap U_1 = W \cap U_2 = \{0\}$ and $U_1 + U_2 = V$. (In fact, it is a direct sum.) So LHS becomes $W \cap (U_1 + U_2) = W \cap V = W$ but RHS becomes $(W \cap U_1) + (W \cap U_2) = \{0\} + \{0\} = \{0\}$.
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> If, additionally, one of the subspaces is contained in W (for example, $U_1 \subseteq W$), the assertion would then hold:
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> Let $v \in (W \cap U_1) + (W \cap U_2)$. Then $v = u_1 + u_2$ for some $u_1 \in W \cap U_1 \subseteq U_1$, $u_2 \in W \cap U_2 \subseteq U_2$. So $v = u_1 + u_2 \in U_1 + U_2$. Also, $u_1, u_2 \in W$, so $v = u_1 + u_2 \in W$. Therefore $v \in W \cap (U_1 + U_2)$. As $v \in (W \cap U_1) + (W \cap U_2)$ is arbitrary, $(W \cap U_1) + (W \cap U_2) \subseteq W \cap (U_1 + U_2)$. (This direction holds in general.)
>
> On the other, let $v \in W \cap (U_1 + U_2)$. Then $v \in W$ and $v \in U_1 + U_2$. Let $u_1 \in U_1 \subseteq W, u_2 \in U_2$ such that $v = u_1 + u_2$. Then $u_2 = v - u_1 \in W$, so $u_2 \in W \cap U_2$. Therefore $v = u_1 + u_2$ where $u_1 \in U_1 = W \cap U_1$ and $u_2 \in W \cap U_2$. Hence $v \in (W \cap U_1) + (W \cap U_2)$. Since v is arbitrary, $W \cap (U_1 + U_2) \subseteq (W \cap U_1) + (W \cap U_2)$.
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> Since $(W \cap U_1) + (W \cap U_2) \subseteq W \cap (U_1 + U_2)$ and $W \cap (U_1 + U_2) \subseteq (W \cap U_1) + (W \cap U_2)$, we must have $W \cap (U_1 + U_2) = (W \cap U_1) + (W \cap U_2)$.

2. Let $U = \{(x, y, z) \in \mathbb{R}^3 : x + 2y + 3z = 0\}$ and $W = \{(x, y, z) \in \mathbb{R}^3 : \frac{x}{4} = \frac{y}{5} = \frac{z}{6}\}$ be subsets of the real vector space $V = \mathbb{R}^3$. Determine if $V = U \oplus W$.

> **Solution:** First of all, we must verify that both U and W are subspaces of V, as they are only given as subsets (not subspaces) in the question. The verification is straight forward and is omitted here.
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> Let U and W be two subspaces of a vector space V, we say that V is the direct sum of U and W provided that

1. $U \cap W = \{0\}$
2. $U + W = V$.

And we denote $V = U \oplus W$.

To investigate the properties of U and W, we can represent the vectors in the subspaces with parameters. One set of parametrizations is as follows:

$$U = \{(-2a - 3b, a, b) \in \mathbb{R}^3 : a, b \in \mathbb{R}\}$$

$$W = \{(4t, 5t, 6t) \in \mathbb{R}^3 : t \in \mathbb{R}\}$$

Solving the system

$$\begin{cases} -2a - 3b = 4t \\ a = 5t \\ b = 6t \end{cases}$$

we can see that $U \cap W = \{0\}$.

It is only left to determine if $U + W = V$. One approach is to show (e.g. with Gauss elimination from MATH1030) that the system

$$\begin{cases} -2a - 3b + 4t = x \\ a + 5t = y \\ b + 6t = z \end{cases}$$

always has a solution $(a, b, t) = \left(\frac{-5x+22y-15z}{32}, \frac{-3x-6y+7z}{16}, \frac{x+2y+3z}{32}\right)$ for arbitrary $(x, y, z) \in \mathbb{R}^3$. The detailed computation is omitted here.

Note

Readers may also prove (and generalize) the follow proposition:

In the real plane $\mathbb{R}^2$, if U is a plane that contains the origin, and W is a straight line passing through the origin and not contained in U, we always have $\mathbb{R}^2 = U \oplus W$.

3. Show that $C([-1,1]) = \{\text{odd function in } C([-1,1])\} \oplus \{\text{even function in } C([-1,1])\}$.

   Similarly, show that $M_{n\times n}(\mathbb{R}) = \mathrm{Sym}_{n\times n} \oplus \mathrm{Skew}_{n\times n}$ where $\mathrm{Sym}_{n\times n} = \{A \in M_{n\times n}(\mathbb{R}) : A^T = A\}$, $\mathrm{Skew}_{n\times n} = \{A \in M_{n\times n}(\mathbb{R}) : A^T = -A\}$

Solution: For the first part, recall that a real-value function f on $[-1,1]$ is odd if $f(-x) = -f(x)$ for every $x \in [-1,1]$, and is even if $f(-x) = f(x)$ for every $x \in [-1,1]$. To show the direct sum, one must first show that the two subsets $\mathcal{O} = \{\text{odd function in } C([-1,1])\}$ and $\mathcal{E} = \{\text{even function in } C([-1,1])\}$ are indeed subspaces of $C([-1,1])$. (Proof omitted.)

If a (continuous) function f on $[-1,1]$ is both odd and even, then we must have $f(x) = f(-x) = -f(x)$ for all $x \in [-1,1]$, which implies f is identically zero. This implies that $\mathcal{O} \cap \mathcal{E} = \{0\}$.

For each function $f \in C([-1,1])$, we define $f_{\text{even}}(x) = \frac{1}{2}(f(x) + f(-x))$, $f_{\text{odd}}(x) = \frac{1}{2}(f(x) - f(-x))$ for $x \in [-1,1]$. Then $f_{\text{even}} \in \mathcal{E}$, $f_{\text{odd}} \in \mathcal{O}$, and $f = f_{\text{even}} + f_{\text{odd}}$. So we have $\mathcal{E} + \mathcal{O} = C([-1,1])$.

From the two results, we have $C([-1,1]) = \{\text{odd function in } C([-1,1])\} \oplus \{\text{even function in } C([-1,1])\}$.

For the second part, notice the similarity between even functions and symmetric matrices and between odd functions and skew-symmetric matrices. Then the same idea can be applied.