2020 B

Week 6 (Feb 16)

- mass, first moments, and center of mass.

Just have to memorize the definitions.

Let Ω be a solid in $\mathbb{R}^3$ with density $\delta(x,y,z)$.

mass $$M = \iiint_{\Omega} \delta \, dV.$$

Frist moment w.r.t. yz-plane

$$M_{yz} = \iiint_{\Omega} x \, \delta \, dV,$$

First moment w.r.t. xz-plane.

$$M_{xz} = \iiint_{\Omega} y \, \delta \, dV,$$

First moment w.r.t. xy-plane

$$M_{xy} = \iiint_{\Omega} z \, \delta \, dV.$$

Center of mass of the solid Ω:

$$\vec{c} = (\bar{x}, \bar{y}, \bar{z}),$$

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}.$$

When $\delta = \text{const}$, center of mass is called the centroid of Ω.

The motivation of the center of mass is to find a point $(\bar{x}, \bar{y}, \bar{z})$ such that the first moments vanish when using $(\bar{x}, \bar{y}, \bar{z})$ as the new origin. It becomes the conditions

$$\iiint_\Omega (x-\bar{x})\,\delta\, dV = 0, \quad \iiint_\Omega (y-\bar{y})\,\delta\, dV = 0, \quad \iiint_\Omega (z-\bar{z})\,\delta\, dV = 0$$

which yield the definition.

For $D \subset \mathbb{R}^2$, we have

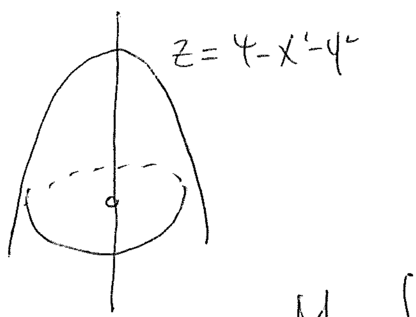
$$M = \iint_D \delta\, dA,$$

$$M_x = \iint_D y\,\delta\, dA$$

$$M_y = \iint_D x\,\delta\, dA$$

$$\vec{C} = (\bar{x}, \bar{y}), \quad \bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}.$$

e.g. Let Ω be the solid bounded by $z = 4 - x^2 - y^2$ over the xy-plane. Find its centroid.



Ω: (x, y, z), $0 \le z \le 4 - x^2 - y^2$

$(x,y) \in D$ where

D is the disk with radius 2.

$$M = \iiint_\Omega \delta\, dV = \delta \iint_D \int_0^{4-x^2-y^2} 1\, dz\, dA = \delta \int_0^{2\pi} \int_0^2 (4-r^2)\, r\, dr\, d\theta$$

$$= 8\pi\delta.$$

$$M_{xy} = \iiint_{\Omega} z\delta \, dV = \delta \iint_{D} \int_0^{4-x^2-y^2} z \, dz \, dA(x,y)$$

$$= \delta \int_0^{2\pi} \int_0^2 \frac{1}{2}(4-r^2)^2 r \, dr \, d\theta$$

$$\vdots$$

$$= \frac{32\pi}{3}\delta$$

$$\therefore \bar{z} = \frac{4}{3}\pi .$$

By symmetry, $\iiint_{\Omega} y\delta \, dV = \iiint_{\Omega} x\delta \, dV = 0$, $\bar{x} = \bar{y} = 0$.
(see Ex 5)

$$\vec{C} = (0, 0, \frac{4}{3}\pi) .$$

- Moments of Inertia.

A particle rotating around an axis has kinetic energy

$$\frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 r^2$$

where ω is the angular speed of the axis and r its distance to the axis. For a solid, the kinetic energy becomes

$$\frac{1}{2}\omega^2 \iiint_{\Omega} r^2(x,y,z)\, \delta(x,y,z)\, dV .$$

Let L be an axis. Define the moment of inertia w.r.t. L:

$$I_L = \iiint_{\Omega} r^2(x,y,z)\, \delta(x,y,z)\, dV ,$$ where

$r(x,y,z)$ is the distance from (x,y,z) to L.

Taking L to be the x-axis,

$$d(x,y,z) = \sqrt{y^2+z^2}, \text{ so}$$

$$I_x = \iiint_\Omega (y^2+z^2)\,\delta\,dV.$$

Similarly,

$$I_y = \iiint_\Omega (x^2+z^2)\,\delta\,dV$$

$$I_z = \iiint_\Omega (x^2+y^2)\,\delta\,dV.$$

When $n=2$,

$$I_x = \iint_D y^2\,\delta\,dA$$

$$I_y = \iint_D x^2\,\delta\,dA$$

and the moment of inertia w.r.t. the origin is

$$I_0 = \iint_D (x^2+y^2)\,\delta\,dA.$$

e.g. Find the moment of inertia for $\Omega = [-a/2, a/2]\times[-b/2, b/2]\times[-c/2, c/2]$
$\delta = \text{Const}$.

$$I_x = \iiint_\Omega (y^2+z^2)\,\delta\,dV$$

$$= \delta \int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\int_{-c/2}^{c/2} (y^2+z^2)\,dz\,dy\,dx$$

$$= 8\delta \int_0^{a/2}\int_0^{b/2}\int_0^{c/2} (y^2+z^2)\, dz\, dy\, dx$$

$$\vdots$$

$$= \frac{abc\delta}{12}(b^2+c^2).$$

Similarly,

$$I_y = \frac{abc\delta}{12}(a^2+c^2), \quad I_z = \frac{abc\delta}{12}(a^2+b^2).$$

- Cylindrical Coordinates

So far, we consider regions of the form

$$\Omega = \left\{ (x,y,z) : \begin{array}{l} f_1(x,y) \le z \le f_2(x,y) \\ (x,y) \in D \end{array} \right\}$$

where D is some region in $\mathbb{R}^2$. Fubini's then becomes

$$\iiint_\Omega f\, dV = \iint_D \int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\, dz\, dA(x,y)$$

In case D can be described as

$$\left\{ (x,y) : \begin{array}{l} x = r\cos\theta, \ y = r\sin\theta \\ r_1(\theta) \le r \le r_2(\theta), \ \theta_1 \le \theta \le \theta_2 \end{array} \right\}$$

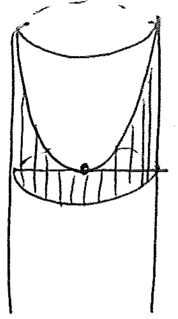
then

$$\iiint_\Omega f\, dV = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\, dz\, r\, dr\, d\theta, \quad \text{where}$$

$x = r\cos\theta$, $y = r\sin\theta$.

The representation of (x,y,z) by (r,θ,z), $r \ge 0$, $\theta \in [0,2\pi)$ or $(-\pi,\pi]$, $z \in \mathbb{R}$, is called the cylindrical coordinate of (x,y,z).

e.g. Find the centroid of the solid Ω which is bounded by $z = x^2+y^2$ and the cylinder $x^2+y^2=4$ over x-y-plane.



Ω: $0 \leq z \leq r^2$
$0 \leq \theta \leq 2\pi$
$0 \leq r \leq 2$

$$\therefore \quad M = \iiint_{\Omega} 1 \, dV = \int_0^{2\pi}\int_0^{2}\int_0^{r^2} dz \, r dr d\theta$$
$$= \int_0^{2\pi}\int_0^{2} r^3 dr d\theta$$
$$= 8\pi.$$

$$M_{xy} = \iiint_{\Omega} z \, dV = \int_0^{2\pi}\int_0^{2}\int_0^{r^2} z dz \, r dr d\theta$$
$$= \int_0^{2\pi}\int_0^{2} \frac{r^4}{2} r dr d\theta$$
$$\doteq \frac{32\pi}{3}$$

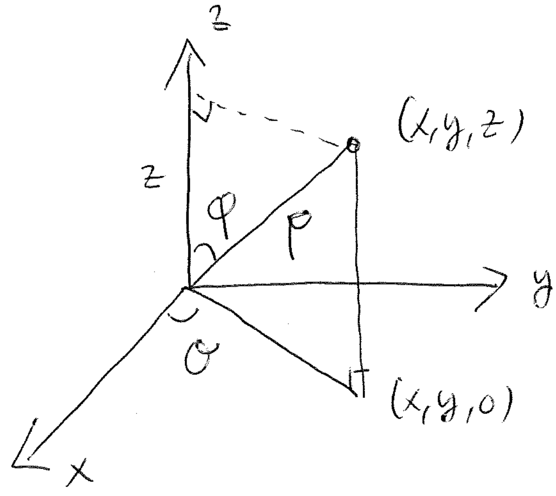
$\therefore \quad \bar{z} = \dfrac{M_{xy}}{M} = \dfrac{4}{3}\pi$. By symmetry, $\bar{x} = \bar{y} = 0$.

$\therefore$ Centroid $\vec{c} = (0, 0, \frac{4}{3}\pi)$.

- Spherical coordinates

Every point $(x,y,z) \neq (0,0,0)$ can be uniquely represented by (ρ, φ, θ), where

$\rho > 0$, $\varphi \in [0, \pi]$, $\theta \in [0, 2\pi)$ (or $(-\pi, \pi]$).



The relations are

$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi .$$

For a region Ω described by

$$\left\{ (x,y,z) : \begin{array}{c} \rho_1(\varphi,\theta) \le \rho \le \rho_2(\varphi,\theta) \\ (\varphi,\theta) \in D \end{array} \right\}$$

$$\iiint_\Omega f\,dV = \iint_D \int_{\rho_1(\varphi,\theta)}^{\rho_2(\varphi,\theta)} f(x,y,z)\ \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

when D is $\{\theta_1 \le \theta \le \theta_2,\ \varphi_1 \le \varphi \le \varphi_2\}$ (a rectangle)

$$\iiint_\Omega f\,dV = \int_{\theta_1}^{\theta_2} \int_{\varphi_1}^{\varphi_2} \int_{\rho_1(\varphi,\theta)}^{\rho_2(\varphi,\theta)} f(x,y,z)\ \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta .$$

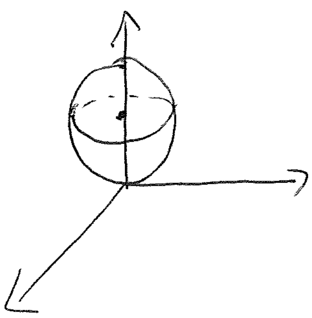
(The formula will be explained later)

e.g. Describe the regions in spherical coordinates.

(a) bdd by $x^2+y^2+(z-1)^2=1$

(b) bdd by $z=\sqrt{x^2+y^2}$ and $z=1$.

(a)



$x^2+y^2+(z-1)^2=1$ describes the sphere of radius 1 center at $(0,0,1)$

$$(\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2 + (\rho\cos\varphi - 1)^2 = 1$$

$$\rho^2\sin^2\varphi + \rho^2\cos^2\varphi - 2\rho\cos\varphi + 1 = 1$$

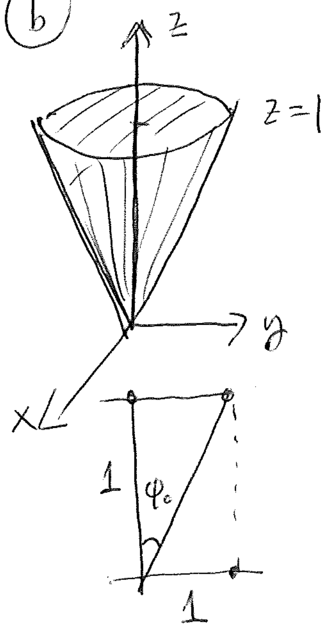
$$\rho^2 - 2\cos\varphi = 0$$

$$\therefore\ \rho = 2\cos\varphi .$$

$$\Omega = \{(x,y,z) : \begin{array}{l} 0 \le \rho \le 2\cos\varphi \\ 0 \le \varphi \le \pi/2,\ 0 \le \theta \le 2\pi \end{array} \}$$

(when $\varphi \in (\frac{\pi}{2}, \pi]$, $\cos\varphi < 0$ No good)

(b)



Ω is the solid cone.

$0 \le \rho \le \rho_2$ where ρ_2 is the plane $z=1$

$$\rho_2 \cos\varphi = 1 \Rightarrow \rho_2 = \frac{1}{\cos\varphi}$$

also

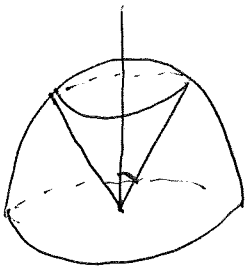
$$\tan\varphi_0 = 1/1 = 1,\ \varphi_0 = \pi/4$$

$\therefore$

$$\Omega = \{(x,y,z) : 0 \le \rho \le \frac{1}{\cos\varphi},\ 0 \le \varphi \le \pi/4,\ 0 \le \theta \le 2\pi\}.$$

e.g. Find the volume and the moment of inertia about z-axis of $\Omega = \{(x,y,z) : 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi/3\}$. Take $\delta = 1$.

Ω is a solid ice-cream cone.



$\varphi_0 = \pi/3$

$$\text{vol} = \iiint_\Omega 1\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \frac{1}{3}\sin\varphi\, d\varphi\, d\theta$$

$$\vdots$$

$$= \pi/3.$$

$$I_z = \iiint (x^2+y^2)\,dV$$

$$= \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 (\rho^2 \sin^2\varphi \cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta)\, \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^4 \sin^3\varphi \, d\rho\, d\varphi\, d\theta$$

$$\vdots$$

$$= \pi/12 .$$

e.g. Let Ω be the region bounded above by $z = 4 - x^2 - y^2$ and the plane $z = 2$. Express

$$\iiint_\Omega f\,dV$$

in all three coordinates.

the plane $z = 2$ and $z = 4 - x^2 - y^2$ cut at

$$2 = 4 - x^2 - y^2,$$

$$x^2 + y^2 = 2,$$

ie, the region is over the disk of radius $\sqrt{2}$.

In rectgl coordinate $\Omega = \{ (x,y,z) \; ; \; 2 \le z \le 4 - x^2 - y^2, \; (x,y) \in D_{\sqrt{2}} \}$

$$\iiint_\Omega f\,dV = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_2^{4-x^2-y^2} f(x,y,z)\,dz\,dy\,dx .$$

In cylindrical coorth,

$$\iiint_\Omega f\,dV = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_2^{4-r^2} f(r\cos\theta, r\sin\theta, z)\,dz\, r\,dr\,d\theta .$$

In spherical coordinates,

$$z=2 \longleftrightarrow \rho=\frac{2}{\cos\varphi},$$

$$z=4-x^2-y^2 \longleftrightarrow \rho\cos\varphi=4-\rho^2\sin^2\varphi,$$

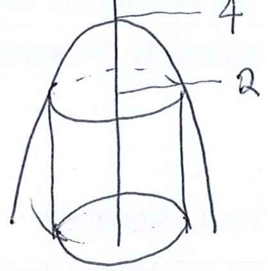
$$\rho^2\sin^2\varphi+\rho\cos\varphi-4=0$$

$$\rho=\frac{-\cos\varphi\pm\sqrt{\cos^2\varphi+16\sin^2\varphi}}{2\sin^2\varphi}$$

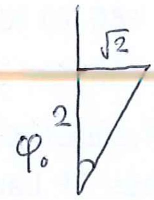
$$=\frac{-\cos\varphi\pm\sqrt{1+15\sin^2\varphi}}{2\sin^2\varphi}$$

$$\rho>0 \Rightarrow \rho=\frac{-\cos\varphi+\sqrt{1+15\sin^2\varphi}}{2\sin^2\varphi}$$

$$\Omega=\left\{(x,y,z):\ \frac{2}{\cos\varphi}\le\rho\le\frac{-\cos\varphi+\sqrt{\cdots}}{2\sin^2\varphi},\ 0\le\varphi\le\varphi_0,\ 0\le\theta\le 2\pi\right\}$$



$$\therefore \iiint_\Omega f\,dV=\int_0^{2\pi}\int_0^{\varphi_0}\int_{2/\cos\varphi}^{\frac{-\cos\varphi+\sqrt{\cdots}}{2\sin^2\varphi}} f(\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \quad \#$$

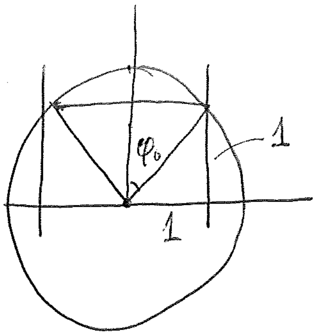


$$\tan\varphi_0=\frac{\sqrt{2}}{2}$$

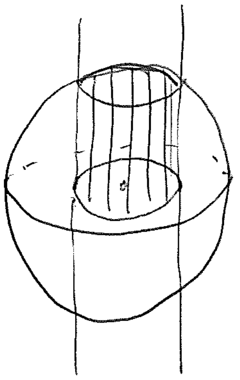
$$\varphi_0=\tan^{-1}\frac{\sqrt{2}}{2}$$

Sometimes, a region could become rather complicated in spherical coordinates.

e.g. Let Ω be the solid bdd by $x^2+y^2+z^2=2$, $x^2+y^2=1$, and $z\ge 0$. Describe it in spherical coordinates.



cross section



$\tan \varphi_0 = 1/1 = 1$

$\varphi_0 = \pi/4$

$\Omega = \Omega_1 \cup \Omega_2$, where

$\Omega_1 = \{(x,y,z) : 0 \leq \rho \leq 1, \ 0 \leq \varphi \leq \pi/4, \ 0 \leq \theta \leq 2\pi\}$

$\Omega_2 = \{(x,y,z) : 0 \leq \rho \leq 1/\sin\varphi, \ \pi/4 \leq \varphi \leq \pi/2, \ 0 \leq \theta \leq 2\pi\}$