2020 B

March 23

Recall, for a v.f.

$$\vec{F} = M\hat{i} + N\hat{j} + P\hat{k} \quad (n=3)$$
$$= M\hat{i} + N\hat{j} \quad (n=2)$$

(I) $\vec{F}$ conservative (gradient) v.f. if

$$\vec{F} = \nabla f, \quad f \text{ potential}$$

(II) $\vec{F}$ independent of path if

$\int_C \vec{F}\cdot d\vec{r}$ only depends on the endpts.

(III) $\vec{F}$ loop property if $\oint_C \vec{F}\cdot d\vec{r} = 0$, $\forall$ closed curve C

$$\text{(I)} \underset{\text{need to prove}}{\Longleftrightarrow} \text{(II)} \underset{\text{easy}}{\Longleftrightarrow} \text{(III)}$$

Moreover, when $\vec{F}$ is conservative,

$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A), \text{ when } C \text{ connects pt } A \text{ to pt } B.$$

Next, a criterion for $\vec{F}$ to be conservative. It is the component test

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y} \qquad (n=3)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad (n=2).$$

REMEMBER the component test is a necessary condition for conservative. There are v.f. passes the test but is NOT conservative.

e.g. $$\vec{F} = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j} + 0\hat{k} \qquad (n=2,3)$$

is one of such examples.

In assignment, 2 more results are given

(1) $\vec{F} = (F_1, F_2, \dots, F_n)$ a v.f. in $\Omega \subseteq \mathbb{R}^n$. The component test becomes
$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}, \quad i,j = 1,\dots n.$$

(2) $\vec{F}$ is a smooth v.f. in a star-shaped region. Then $\vec{F}$ satisfies the component test
$$\Rightarrow \vec{F} \text{ has a potential.}$$

Thus, whether the component test is sufficient for the existence of a potential depends on the geometry of the region $\vec{F}$ is defined.

For instance, $n=2$, the $\vec{F}$ in the example above is defined in
$$D = \{ (x,y) : (x,y) \neq (0,0) \}$$
which is clearly not star-shaped.

Return to today lecture.

An expression of the form

$$M\,dx + N\,dy + P\,dz$$

is called a differential form. It is exact if

$$\frac{\partial f}{\partial x} = M, \quad \frac{\partial f}{\partial y} = N, \quad \frac{\partial f}{\partial z} = P$$

for some fcn f. Clearly, a differential form is exact if and only if $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ has potential f.

e.g. Evaluate

$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz.$$

Implicitly it means this diff. form is exact.

$$\frac{\partial f}{\partial x} = y \Rightarrow f(x,y,z) = xy + g(y,z)$$

$$\frac{\partial f}{\partial y} = x \Rightarrow x + \frac{\partial g}{\partial y} = x \Rightarrow g(y,z) = h(z)$$

$$\frac{\partial f}{\partial z} = 4 \Rightarrow 0 + h_z(z) = 4 \Rightarrow h(z) = 4z + C.$$

(take $C = 0$)

$\therefore\ f(x,y,z) = xy + 4z$

$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz = f(2,3,-1) - f(1,1,1)$$

$$= 2\times 3 + 4\times(-1) - [1\times 1 + 4]$$

$$= 2 - 5$$

$$= -3 \ \#$$

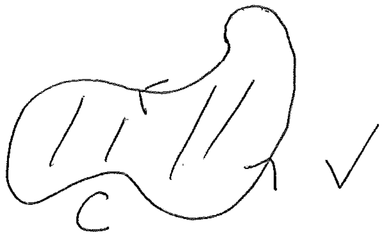
16.4 Green's theorem.

Green's theorem is a formula relating line integrals to double integrals.

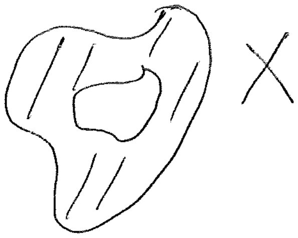
The setting is:

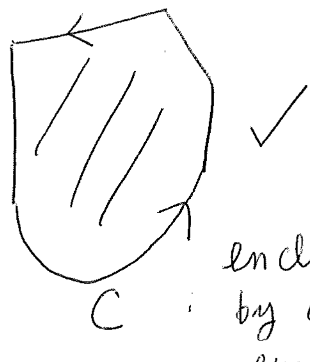
$\vec{F} = M\hat{i} + N\hat{j}$ a smooth v.f. in D,

D is region enclosed by a single, piecewise smooth simple, closed curve C.

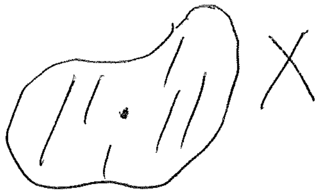


enclosed by 2 curves





enclosed by 2 curves (the interior one degenerate into a pt)



enclosed by a curve

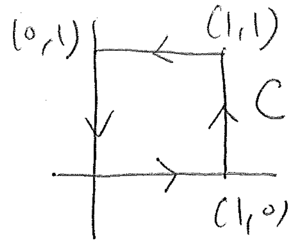
<u>Theorem</u> Setting as above,

$$\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

when C is in anti-clockwise direction.

e.g. Evaluate

$\oint_C xy\,dy - y^2\,dx$ when C is the square.



Instead of doing 4 line integrals, we apply Green's thm, here

$$M = -y^2, \quad N = xy$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = y - (-2y) = 3y$$

$$\therefore \oint_C xy\,dy - y^2\,dx = \iint_D 3y\,dA$$

$$= \int_0^1 \int_0^1 3y^2\,dy\,dx = 3/2 \quad \#$$