Solution 9

Supplementary Problems

1. Let $F = (F_1, \cdots, F_n)$ be a smooth vector field in an open region in $\mathbb{R}^n$. Show that if it is conservative, then the necessary conditions hold

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}\ , \quad \forall i, j.$$

Solution. Let $F = \nabla f$. Then

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_i},$$

and

$$\frac{\partial F_j}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{\partial f}{\partial x_j},$$

so they are equal. When $n = 3$, this reduces to the usual compatibility conditions (or necessary conditions, or component test):

$$M_z = P_x, \quad M_y = N_x, \quad N_z = P_y\ .$$

2. A region is called star-shaped if there is a point O inside so that the line segment connecting any point in this region to O lies completely in this region. For simplicity take O to be the origin.

   (a) Show that in case the vector field $\mathbf{F}$ admits a potential g in this region, then

$$g(x, y, z) = \int_0^1 \mathbf{F}(tx, ty, tz) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\, dt\ .$$

   (b) Show that when $\mathbf{F}$ passes the component test, the above formula defines a potential function for $\mathbf{F}$.

Solution. (a) We will work on the general dimension, so $\mathbf{F} = (F_1, F_2, \cdots, F_n)$. By Chain Rule we have

$$\begin{aligned}\frac{dg}{dt}(tx_1, tx_2, \cdots, tx_n) &= \frac{\partial g}{\partial x_1}(t\mathbf{x})x_1 + \frac{\partial g}{\partial x_2}(t\mathbf{x})x_2 + \cdots + \frac{\partial g}{\partial x_n}(t\mathbf{x})x_n \\ &= F_1(t\mathbf{x})x_1 + F_2(t\mathbf{x})x_2 + \cdots + F_n(t\mathbf{x})x_n\ .\end{aligned}$$

Therefore,

$$\begin{aligned}g(x, y) - g(0, 0) &= \int_0^1 \frac{dg}{dt}(tx_1, tx_2, \cdots, tx_n)\, dt \\ &= \int_0^1 F_1(t\mathbf{x})x_1 + F_2(t\mathbf{x})x_2 + \cdots + F_n(t\mathbf{x})x_n\, dt \\ &= \int_0^1 \mathbf{F}(t\mathbf{x}) \cdot \mathbf{x}\, dt\ .\end{aligned}$$

We are done after setting $g(0, 0) = 0$.

(b) In a general dimension, the component test becomes

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} ,$$

for different $i, j = 1, 2, \cdots, n$. With the above formula for g,

$$\begin{aligned}
\frac{\partial g}{\partial x_i} &= \int_0^1 [\frac{\partial F_1}{\partial x_i}(t\mathbf{x})tx_1 + \frac{\partial F_2}{\partial x_i}(t\mathbf{x})tx_2 + \cdots + \frac{\partial F_n}{\partial x_i}(t\mathbf{x})tx_n + F_i(t\mathbf{x})]\, dt \\
&= \int_0^1 [\frac{\partial F_i}{\partial x_1}(t\mathbf{x})tx_1 + \frac{\partial F_i}{\partial x_2}(t\mathbf{x})tx_2 + \cdots + \frac{\partial F_i}{\partial x_n}(t\mathbf{x})tx_n + F_i(t\mathbf{x})]\, dt \\
&= \int_0^1 \frac{d}{dt} tF_i(t\mathbf{x})\, dt \\
&= F_i(t\mathbf{x})\Big|_0^1 \\
&= F_i(\mathbf{x}) .
\end{aligned}$$