# Solution to Assignment 1

1. Consider the function $H$ in $\mathbb{R}^2$ defined by $H(x,y)=1$ whenever $x,y$ are rational numbers and equals to 0 otherwise. Show that $H$ is not integrable in any rectangle.

   **Solution.** Let $P$ be any partition of the rectangle. By choosing tags points $(x^*,y^*)$ where $x^*$ and $y^*$ are rational numbers,

$$\sum_{j,k} H(x_j^*,y_k^*)\Delta x_j \Delta y_k = \sum_{j,k} \Delta x_j \Delta y_k$$

   which is equal to the area of $R$. On the other hand, by choosing the tags so that $x^*$ is irrational, $H(x^*,y^*)=0$ so that

$$\sum_{j,k} H(x_j^*,y_k^*)\Delta x_j \Delta y_k = \sum_{j,k} 0 \times \Delta x_j \Delta y_k = 0 \ .$$

   Depending the choice of tags, the Riemann sums are not the same for the same partition, hence they cannot tend to the same limit. We conclude that $H$ is not integrable.

2. Give an example of a nonnegative, integrable function which does not vanish identically and yet

$$\iint_R f\, dA = 0 \ .$$

   **Solution.** Let $R=[a,b]\times[c,d]$ and set $f(x,y)=0$ in $R$ except $f(a,b)=1$. $f$ is not identically to 0. As $f$ is piecewise continuous (it is discontinuous at the origin only), it is integrable. By choosing a sequence of partitions $P$, $\|P\|\to 0$, and tag points at the center, $R(f,P)=0$, so

$$\iint_R f\, dA = 0 \ .$$

3. Let $f$ be a nonnegative, continuous function on $R$. Show that

$$\iint_R f\, dA = 0 \ ,$$

   implies that $f$ vanishes identically.

   **Solution.** It is equivalent to proving: Whenever $f$ is a nonnegative, continuous function which does not vanish identically, its integral is positive. For, let $(x_0,y_0)\in R$ such that $f(x_0,y_0)>0$. By continuity, we can find a small rectangle $R_0$ containing $(x_0,y_0)$ such that $f(x,y)\ge f(x_0,y_0)/2$ for all $(x,y)\in R_0$. Define a new function $g$ by $g(x,y)=f(x_0,y_0)/2,\ (x,y)\in R_0$ and $g(x,y)=0$ elsewhere. Then $g$ is integrable and $f\ge g$ in $R$. It follows that

$$\iint_R f \ge \iint_R g = \frac{1}{2} f(x_0,y_0)|R_0| > 0 \ ,$$

   done.