MATH1520AB 2021-22 Tutorial 3 (week 5)

1. Suppose that

$$f(x) = \begin{cases} ax + b & \text{if } x < 0; \\ \sin x + 3 & \text{if } x \geq 0 \end{cases}$$

where a and b are real numbers.

Given that f is differentiable at $x = 0$, find the values of a and b. (Use $\lim_{x\to 0} \frac{\sin x}{x} = 1$)

Answer.

Since f is differentiable at $x = 0$, f is continuous at $x = 0$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} ax + b = a(0) + b = b$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sin x + 3 = \sin(0) + 3 = 3$$

So, $b = 3$.

Since f is differentiable at $x = 0$, $Lf'(0) = Rf'(0)$. Note that $f(0) = \sin(0) + 3 = 3$.

$$Lf'(0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{ax + 3 - 3}{x} = \lim_{x\to 0^-} a = a$$

$$Rf'(0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{\sin x + 3 - 3}{x} = \lim_{x\to 0^-} \frac{\sin x}{x} = 1$$

So, $a = 1$.

2. Find the derivative of $f(x) = \sqrt{2x^2 - 1}$ by first principle and by chain rule.

Answer. By first principle,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{\sqrt{2(x+h)^2 - 1} - \sqrt{2x^2 - 1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2(x+h)^2 - 1} - \sqrt{2x^2 - 1}}{h} \frac{\sqrt{2(x+h)^2 - 1} + \sqrt{2x^2 - 1}}{\sqrt{2(x+h)^2 - 1} + \sqrt{2x^2 - 1}} \\
&= \lim_{h\to 0} \frac{(2(x+h)^2 - 1) - (2x^2 - 1)}{h(\sqrt{2(x+h)^2 - 1} + \sqrt{2x^2 - 1})} \\
&= \lim_{h\to 0} \frac{4xh + h^2}{h(\sqrt{2(x+h)^2 - 1} + \sqrt{2x^2 - 1})} \\
&= \lim_{h\to 0} \frac{4x + h}{\sqrt{2(x+h)^2 - 1} + \sqrt{2x^2 - 1}} \\
&= \frac{2x}{\sqrt{2x^2 - 1}}
\end{aligned}$$

By chain rule,

$$f'(x) = \frac{1}{2}(2x^2-1)^{-\frac{1}{2}}\frac{d}{dx}(2x^2-1) = \frac{4x}{2\sqrt{2x^2-1}} = \frac{2x}{\sqrt{2x^2-1}}$$

3. Find the derivative of the following functions.

   (a) $f(x) = 2^{-x^2+13x-42}$

   (b) $f(x) = \dfrac{1}{\ln(\sqrt{5-x})}$

   Answer.

   (a)

$$f(x) = 2^{-x^2+13x-42} = e^{(-x^2+13x-42)\ln 2}$$

$$f'(x) = e^{(-x^2+13x-42)\ln 2}\frac{d}{dx}[(-x^2+13x-42)\ln 2]$$

$$f'(x) = 2^{-x^2+13x-42}(\ln 2)(-2x+13)$$

   (b)

$$f(x) = \frac{1}{\ln(\sqrt{5-x})}$$

$$f'(x) = (-\frac{1}{[\ln(\sqrt{5-x})]^2})(\frac{1}{\sqrt{5-x}})(\frac{1}{2\sqrt{5-x}})(-1)$$

$$f'(x) = \frac{1}{2(5-x)[\ln(\sqrt{5-x})]^2} = \frac{2}{(5-x)[\ln(5-x)]^2}$$

4. If $5x^3f(x)^2 - 7xf(x) = 9$, find $f'(x)$ in terms of $f(x)$.

   Answer. Differentiating both sides of $5x^3f(x)^2 - 7xf(x) = 9$, we have

$$\frac{d}{dx}(5x^3f(x)^2) - \frac{d}{dx}(7xf(x)) = \frac{d}{dx}9$$

   By product rule,

$$(\frac{d}{dx}5x^3)f(x)^2 + 5x^3\frac{d}{dx}f(x)^2 - (\frac{d}{dx}7x)f(x) - 7x\frac{d}{dx}f(x) = 0$$

$$15x^2f(x)^2 + 5x^3\frac{d}{dx}f(x)^2 - 7f(x) - 7x\frac{d}{dx}f(x) = 0$$

   By chain rule,

$$15x^2f(x)^2 + 5x^3[2f(x)f'(x)] - 7f(x) - 7xf'(x) = 0$$

$$15x^2f(x)^2 - 7f(x) + 10x^3f(x)f'(x) - 7xf'(x) = 0$$

$$f'(x) = \frac{7f(x) - 15x^2f(x)^2}{10x^3f(x) - 7x}$$