MATH1520AB 2021-22 Quiz 2 (week 4) Solution

Full marks: 10 marks

Time allowed: 15 minutes

1. Evaluate the following limits (without using the L'Hopital's rule).

   (a) $\lim_{x\to\infty} \frac{x^2+3x+1}{2x^2+5}$

   (b) $\lim_{x\to 0} \frac{\sqrt{x^2+2}-2}{x}$

   (c) $\lim_{x\to -1} \frac{|x|-1}{x^2-1}$

   Answer.

   (a) $\lim_{x\to\infty} \frac{x^2+3x+1}{2x^2+5} = \lim_{x\to\infty} \frac{1+3/x+1/x^2}{2+5/x^2} = \frac{1+0+0}{2+0} = \frac{1}{2}.$

   (b) $\lim_{x\to 0} \frac{\sqrt{x^2+2}-2}{x} = \lim_{x\to 0} \frac{x^2+2-2}{x(\sqrt{x^2+2}+2)} = \lim_{x\to 0} \frac{x}{\sqrt{x^2+2}+2} = \frac{0}{\sqrt{0+2}+2} = 0.$

   (c) When $x \in (-\infty, 0)\backslash\{-1\}$, $\frac{|x|-1}{x^2-1} = \frac{-x-1}{x^2-1}$. Thus,
   $\lim_{x\to -1} \frac{|x|-1}{x^2-1} = \lim_{x\to -1} \frac{-x-1}{x^2-1} = \lim_{x\to -1} -\frac{(x+1)}{(x+1)(x-1)} = \lim_{x\to -1} -\frac{1}{x-1} = \frac{1}{2}.$

2. For what values of c, d is the following function f continuous on $\mathbb{R}$?

$$f(x) = \begin{cases} x+1, & \text{if } x \le 0 \\ x^2+c, & \text{if } 0 < x \le 3 \\ cx-d, & \text{if } x > 3 \end{cases}$$

   Answer. On subintervals: $(-\infty, 0), (0, 3), (3, +\infty)$, f is a polynomial, so it is continuous.
   $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x+1 = 1$
   $\lim_{x\to 0^+} f(x) = \lim_{x\to 0+} x^2+c = c$
   We need that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 0+1 = 1$
   This implies that $c = 1$.
   $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x^2+c = \lim_{x\to 3^-} x^2+1 = 10$
   $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} cx-d = \lim_{x\to 3^+} x-d = 3-d$
   We need that $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) = 10.$
   This implies that $3-d = 10$, so $d = -7$.