MATH1520 University Mathematics for Applications **Fall 2021**

Chapter 9: Indefinite Integrals

Learning Objectives:
(1) Compute indefinite integrals.
(2) Use the method of substitution to find indefinite integrals.
(3) Use integration by parts to find integrals and solve applied problems.
(4) Explore the antiderivatives of rational functions.

9.1 Antiderivatives

Definition 9.1.1. A function $F(x)$ is called an antiderivative of $f(x)$ if

$$F'(x) = f(x)$$

for every x in the domain of $f(x)$.

Example 9.1.1.

1. $F(x) = \frac{1}{3}x^3+5x+2$ is an antiderivative of $f(x) = x^2+5$, since $F'(x) = (\frac{1}{3}x^3+5x+2)' = x^2 + 5$.
2. e^x is an antiderivative of e^x, since $(e^x)' = e^x$.

Theorem 9.1.1 (Fundamental Property of Antiderivatives). *If $F(x)$ is an antiderivative of $f(x)$, then all antiderivative of $f(x)$ can be written as*

$$F(x) + C, \quad C \text{ is an arbitrary constant.}$$

Proof. 1. For any constant C,

$$(F(x) + C)' = F'(x) = f(x),$$

so, $F(x) + C$ is an antiderivative of $f(x)$.

2. For any antiderivative $G(x)$ with $G'(x) = f(x)$,

$$(G(x) - F(x))' = f(x) - f(x) = 0,$$

then, $G(x) - F(x) = C$ for some constant C.

Thus, the general antiderivative of $f(x)$ is $F(x) + C$, $C \in \mathbb{R}$. □

Definition 9.1.2. The indefinite integral of $f(x)$ is the collection of all antiderivatives of $f(x)$, denoted by

$$\int f(x)\,dx,$$

where $\int$ is the integral symbol, $f(x)$ is the integrand, and dx identifies x as the variable of integration.

The process of finding all antiderivatives is called indefinite integration.

Remark. It is useful to remember that if you have performed an indefinite integration calculation that leads you to believe that $\int f(x)\,dx = G(x) + C$, then you can check your calculation by differentiating $G(x)$:

If $G'(x) = f(x)$, then the integration $\int f(x)\,dx = G(x) + C$ is correct, but if $G'(x)$ is anything other than $f(x)$, you've made a mistake.

$$\boxed{F'(x) = f(x) \quad \Rightarrow \quad \int f(x)\,dx = F(x) + C}$$

The fact that indefinite integration and differentiation are reverse operations, except for the addition of the constant of integration, can be expressed symbolically as

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$$

and

$$\int F'(x)\,dx = F(x) + C.$$

9.2 Basic integration formulas

The relationship between differentiation and antidifferentiation enables us to establish the following integration rules by "reversing" analogous differentiation rules.

Theorem 9.2.1.

1. $\int k\,dx = kx + C$ *for constant* k.
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ *for all* $n \neq -1$
3. $\int \frac{1}{x}\,dx = \ln|x| + C$ *for all* $x \neq 0$.
4. $$\begin{aligned} \int e^x\,dx &= e^x + C, \\ \int a^x\,dx &= \tfrac{1}{\ln a}a^x + C \qquad a > 0, a \neq 1. \end{aligned}$$

Theorem 9.2.2.

1. $\int kf(x)\,dx = k\int f(x)\,dx,$ *(constant multiple rule)*
2. $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx,$ *(sum/difference rule)*

Caution: Both sides of the equality involve constant C.

Example 9.2.1.

1.
$$\begin{aligned} \int 3x^7 dx &= 3\int x^7 dx \\ &= 3 \cdot \frac{x^8}{8} + C. \end{aligned}$$

2.
$$\begin{aligned} \int \frac{1}{\sqrt{x}}\,dx &= \int x^{-1/2}\,dx \\ &= \frac{1}{1/2}x^{1/2} + C. \\ &= 2\sqrt{x} + C \end{aligned}$$

3.

$$\begin{aligned}\int (2x^5 + 8x^3 - 3x^2 + 5)\,dx &= 2\int x^5\,dx + 8\int x^3\,dx - 3\int x^2\,dx + \int 5\,dx \quad \text{(No need to add } C\text{)}\\ &= 2\left(\frac{x^6}{6}\right) + 8\left(\frac{x^4}{4}\right) - 3\left(\frac{x^3}{3}\right) + 5x + C \quad \text{(Add one } C\text{)}\\ &= \frac{1}{3}x^6 + 2x^4 - x^3 + 5x + C.\end{aligned}$$

4.

$$\begin{aligned}\int \left(\frac{x^3 + 2x - 7}{x}\right) dx &= \int \left(x^2 + 2 - \frac{7}{x}\right) dx\\ &= \frac{1}{3}x^3 + 2x - 7\ln|x| + C.\end{aligned}$$

5.

$$\begin{aligned}\int (3e^t + \sqrt{t})\,dt &= \int (3e^t + t^{1/2})\,dt\\ &= 3\left(e^t\right) + \frac{1}{3/2}t^{3/2} + C\\ &= 3e^t + \frac{2}{3}t^{3/2} + C.\end{aligned}$$

Exercise 9.2.1.

$$\int \frac{(x + \sqrt{x})(x + 1)}{\sqrt{x}}\,dx = \frac{2}{5}x^{\frac{5}{2}} + \frac{1}{2}x^2 + \frac{2}{3}x^{\frac{3}{2}} + x + C$$

Example 9.2.2. Find the function $f(x)$ whose tangent line has slope $4x^3 + 5$ for each value of x and whose graph passes through the point (1, 10).

Solution. The slope of the tangent line at each point $(x, f(x))$ is the derivative $f'(x)$. Thus,

$$f'(x) = 4x^3 + 5$$

and so $f(x)$ is the antiderivative

$$\int f'(x)\,dx = \int (4x^3 + 5)\,dx = x^4 + 5x + C.$$

To find C, use the fact that the graph of f passes through $(1, 10)$. That is, substitute $x = 1$ and $f(1) = 10$ into the equation for $f(x)$ and solve for C to get

$$10 = (1)^4 + 5(1) + C \qquad \text{or} \qquad C = 4.$$

Thus, the desired function is $f(x) = x^4 + 5x + 4$. ■

9.3 Integration by Substitution ("reversing" the chain rule)

Motivation

Let $f(x) = (x^2 + 3x - 5)^{10}$. We can compute $f'(x)$ using the Chain Rule. It is:

$$f'(x) = 10(x^2 + 3x - 5)^9 \cdot (2x + 3) = (20x + 30)(x^2 + 3x - 5)^9.$$

Conversely, we have

$$\int (20x + 30)(x^2 + 3x - 5)^9 \, dx = (x^2 + 3x - 5)^{10} + C.$$

How would we obtain this indefinite integral without starting with $f(x)$?

Let $u = x^2 + 3x - 5$. Thus

$$\frac{du}{dx} = 2x + 3, \quad \text{or} \quad du = (2x + 3)dx.$$

Therefore,

$$\begin{aligned}\int (20x + 30)(x^2 + 3x - 5)^9 \, dx &= \int 10(2x + 3)(x^2 + 3x - 5)^9 \, dx \\ &= \int 10(\underbrace{x^2 + 3x - 5}_{u})^9 \, \underbrace{(2x + 3) \, dx}_{du} \\ &= \int 10u^9 \, du \\ &= u^{10} + C \quad \text{(replace } u \text{ with } x^2 + 3x - 5\text{)} \\ &= (x^2 + 3x - 5)^{10} + C\end{aligned}$$

More generally, we have

Theorem 9.3.1 (Integration by Substitution)**.**

$$\boxed{\int f(g(x))g'(x) \, dx \overset{u=g(x)}{=\!=} \int f(u) \, du}$$

Key idea: Make a guess $u = g(x)$, realize the integrand as a product of $f(u)$ and $u'(x)$.

Example 9.3.1.

$$\int (2x + 1)^{2019} dx.$$

Solution. Let $u = g(x) = 2x + 1$, $f(u) = u^{2019}$. Then $du = 2dx$.

$$\begin{aligned}\int (2x+1)^{2019} dx &= \frac{1}{2}\int \underbrace{(2x+1)^{2019}}_{f(g(x))} \cdot \underbrace{2}_{g'(x)} dx \\ &= \frac{1}{2}\int u^{2019}\, du \\ &= \frac{u^{2020}}{2 \times 2020} + C \\ &= \frac{(2x+1)^{2020}}{4040} + C.\end{aligned}$$

Remark: usually, it is more convenient to write:

$$\begin{aligned}\int (2x+1)^{2019} dx &= \int u^{2019} \frac{1}{2} du \qquad (\frac{du}{dx} = 2 \Rightarrow dx = \frac{1}{2} du) \\ &= \frac{u^{2019}}{2 \times 2020} + C \\ &= \frac{(2x+1)^{2020}}{4040} + C.\end{aligned}$$

■

Example 9.3.2. Evaluate $\int \frac{7}{-3x+1}\, dx$.

Solution. Let $u = -3x + 1$, then $\frac{du}{dx} = -3$, $dx = -\frac{1}{3} du$.

$$\begin{aligned}\int \frac{7}{-3x+1}\, dx &= \int \frac{7}{u}\frac{du}{-3} \\ &= \frac{-7}{3}\int \frac{du}{u} \\ &= \frac{-7}{3}\ln|u| + C \\ &= -\frac{7}{3}\ln|-3x+1| + C.\end{aligned}$$

■

Example 9.3.3. Evaluate $\int x\sqrt{x+3}\,dx$.

Solution. Let $u = x+3$, then $x = u-3$, $dx = du$, so,

$$\begin{aligned}\int x\sqrt{x+3}\,dx &= \int (u-3)u^{\frac{1}{2}}\,du\\ &= \int \left(u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\right)du\\ &= \frac{2}{5}u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + C\\ &= \frac{2}{5}(x+3)^{\frac{5}{2}} - 2(x+3)^{\frac{3}{2}} + C.\end{aligned}$$

■

Exercise 9.3.1.

1. $\int \sqrt{3x+1}\,dx = \frac{2}{9}(3x+1)^{\frac{3}{2}} + C$
2. $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C,$ where $a \neq 0$.
3. $\int x(x-1)^{100}\,dx = \frac{1}{102}(x-1)^{102} + \frac{1}{101}(x-1)^{101} + C$

Example 9.3.4. Evaluate $\int xe^{x^2+5}\,dx$

Solution. Let $u = g(x) = x^2+5$, hence $du = 2x\,dx$.

$$du = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}du = x\,dx.$$

We can now substitute.

$$\begin{aligned}\int xe^{x^2+5}\,dx &= \int e^{\overbrace{x^2+5}^{u}}\underbrace{x\,dx}_{\frac{1}{2}du}\\ &= \int \frac{1}{2}e^u\,du\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}e^{u} + C \quad \text{(now replace } u \text{ with } x^2+5\text{)} \\
&= \frac{1}{2}e^{x^2+5} + C.
\end{aligned}
$$

Remark: Sometimes, we even do not need to introduce the new variable u, just keep in mind which part should be regarded as $u = g(x)$.

$$
\begin{aligned}
\int xe^{x^2+5}\,dx &= \int \frac{1}{2}e^{x^2+5}\,d(x^2+5) \quad (\text{Regard } u = x^2+5) \\
&= \frac{1}{2}e^{x^2+5} + C.
\end{aligned}
$$

■

Example 9.3.5. Evaluate $\int x^3\sqrt{x^4+1}\,dx$

Solution.

$$
\begin{aligned}
\int x^3\sqrt{x^4+1}\,dx &= \int \frac{1}{4}\sqrt{x^4+1}\,d(x^4+1) \quad (\text{Regard } u = x^4+1) \\
&= \frac{1}{6}(x^4+1)^{3/2} + C.
\end{aligned}
$$

■

Example 9.3.6. Evaluate $\int \frac{1}{x\ln x}\,dx$

Solution.

$$
\begin{aligned}
\int \frac{1}{x\ln x}\,dx &= \int \frac{1}{\ln x}\,d(\ln x)(\text{Regard } u = \ln x) \\
&= \int \frac{1}{u}\,du \\
&= \ln|u| + C \\
&= \ln|\ln x| + C.
\end{aligned}
$$

Remark: To avoid mistakes, we can take the derivative to verify our answer. ■

Exercise 9.3.2.

1. $\displaystyle\int x^3 e^{x^4}\,dx = \frac{1}{4}e^{x^4} + C.$
2. $\displaystyle\int 6x\sqrt{x^2+3}\,dx = 2(x^2+3)^{\frac{3}{2}} + C.$
3. $\displaystyle\int e^x\sqrt{e^x+1}\,dx = \frac{2}{3}(e^x+1)^{\frac{3}{2}} + C.$
4. $\displaystyle\int (2x-1)(x^2-x)^{100}\,dx = \frac{1}{101}(x^2-x)^{101} + C$

9.4 Integration by Parts ("reversing" the Leibniz rule)

Motivation

Let $u(x)$ and $v(x)$ be differentiable functions. By the product rule, we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

or

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrating both sides with respect to x,

$$\begin{aligned}\int u\frac{dv}{dx}\,dx &= \int \frac{d}{dx}(uv)\,dx - \int v\frac{du}{dx}\,dx \\ &= uv - \int v\frac{du}{dx}\,dx\end{aligned}$$

which is

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

or

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Key Idea: Write the integrand as product of $u(x)$ and $v'(x)$, then integrate by parts.

Example 9.4.1. Compute $\displaystyle\int xe^x\,dx.$

Solution.

$$\begin{aligned}\int xe^x\,dx &= \int x\,de^x \qquad (u = x, v = e^x)\\ &= xe^x - \int e^x\,dx\\ &= xe^x - e^x + C\end{aligned}$$

Question: What happens if we let $u = e^x$ and $v = \frac{1}{2}x^2$?

$$\begin{aligned}\int xe^x\,dx &= \int e^x\,d\left(\frac{1}{2}x^2\right)\\ &= \frac{1}{2}x^2e^x - \int \frac{1}{2}x^2\,de^x\\ &= \frac{1}{2}x^2e^x - \int \frac{1}{2}x^2e^x\,dx \quad \text{(More complicated!)}\end{aligned}$$

■

Example 9.4.2.

$$\begin{aligned}\int x\ln x\,dx &= \int \ln x\,d\left(\frac{1}{2}x^2\right) \qquad (u = \ln x, v = \frac{1}{2}x^2)\\ &= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x^2\,d(\ln x)\\ &= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx\\ &= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C\end{aligned}$$

Question: What happens if we let $\int x\ln x\,dx = \int x\,d(?)$

$v'(x) = \ln x$, not easy to find v!

Remark. Choose proper u and v such that:

1. it's easy to write the integral as $\int u\,dv$;
2. it simplifies the problem after integration by parts.

Exercise 9.4.1.

1. $\int x^2\ln x\,dx = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C$
2. $\int xa^x\,dx = \frac{1}{\ln a}xa^x - \frac{1}{\ln^2 a}a^x + C, \quad (a > 0, a \neq 1)$

Example 9.4.3.

$$\begin{aligned}\int \ln x\,dx &= x\ln x - \int x\,d(\ln x) \qquad (u=\ln x, v=x)\\ &= x\ln x - \int 1\,dx\\ &= x\ln x - x + C\end{aligned}$$

Exercise 9.4.2. $\int \log_a x\,dx = x\log_a x - \dfrac{x}{\ln a} + C$

Hint: either integration by parts directly, or use $\log_a x = \dfrac{\ln x}{\ln a}$.

Example 9.4.4. (Integration by parts twice)

1.

$$\begin{aligned}\int x^2e^x\,dx &= \int x^2\,de^x\\ &= x^2e^x - \int e^x\,dx^2\\ &= x^2e^x - \int 2xe^x\,dx\\ &= x^2e^x - \int 2x\,de^x\\ &= x^2e^x - 2(xe^x - \int e^x\,dx)\\ &= x^2e^x - 2(xe^x - e^x + C)\\ &= x^2e^x - 2xe^x + 2e^x + C'\end{aligned}$$

2.

$$\begin{aligned}\int \ln^2 x\,dx &= x\ln^2 x - \int x\,d(\ln^2 x)\\ &= x\ln^2 x - \int x\cdot 2\ln x\cdot\frac{1}{x}\,dx\\ &= x\ln^2 x - \int 2\ln x\,dx\\ &= x\ln^2 x - 2x\ln x + 2\int x\,d(\ln x)\\ &= x\ln^2 x - 2x\ln x + 2x + C\end{aligned}$$

Exercise 9.4.3. $\int (x^2+2x+3)e^x\,dx = (x^2+3)e^x + C.$

9.5 Integration of Rational Functions

Rational function:

$$R(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials with $q(x) \neq 0$.

How to integrate $\displaystyle\int \frac{p(x)}{q(x)}\,dx$?

9.5.1 $\deg q(x) = 1 : q(x) = ax + b, a \neq 0$

Let $a \neq 0$. By long division,

$$\frac{p(x)}{ax+b} \overset{\text{long division}}{=\!=\!=} \underbrace{A(x)}_{\text{polynomial}} + \underbrace{\frac{r}{ax+b}}_{\text{know how to integrate!}},$$

where $A(x)$ is a polynomial and r is a constants.

$$\int \frac{1}{ax+b}\,dx = \int \frac{1}{ax+b} \cdot \frac{1}{a}\,d(ax+b) = \frac{1}{a}\ln|ax+b| + C$$

Example 9.5.1. Evaluate

$$\int \frac{x^2+3x+5}{x+1}\,dx.$$

Solution. By the long division

$$\begin{array}{r|l} & \quad\quad\; x+2. \\ \hline x+1) & x^2+3x+5 \\ & -x^2 \;\; -x \\ \hline & \quad\;\; 2x+5 \\ & \quad -2x-2 \\ \hline & \quad\quad\quad 3 \end{array}$$

So,

$$\begin{aligned} \int \frac{x^2+3x+5}{x+1}\,dx &= \int (x+2) + \frac{3}{x+1}\,dx \\ &= \frac{x^2}{2} + 2x + 3\ln|x+1| + C. \end{aligned}$$

■

9.5.2 $\deg q(x) = 2 : q(x) = ax^2 + bx + c, a \neq 0$

$$\frac{p(x)}{ax^2+bx+c} \overset{\text{long division}}{=\!=\!=} \underbrace{A(x)}_{\text{polynomial}} + \underbrace{\frac{rx+s}{ax^2+bx+c}}_{\text{our focus!}},$$

3 subcases for $\displaystyle\int \frac{rx+s}{ax^2+bx+c}\,dx$:

$$(i)\ \Delta > 0, \quad (ii)\ \Delta = 0, \quad (iii)\ \Delta < 0. \qquad (\Delta = b^2 - 4ac)$$

case (i) : $\Delta > 0, ax^2 + bx + c = a(x - x_1)(x - x_2), x_1 \neq x_2$

$$\frac{rx+s}{ax^2+bx+c} = \frac{A}{x-x_1} + \frac{B}{x-x_2},$$

which are called partial fractions.

Example 9.5.2. Evaluate

$$\int \frac{5x-7}{x^2-2x-3}dx.$$

Solution. Suppose

$$\frac{5x-7}{(x-3)(x+1)} \equiv \frac{A}{x-3} + \frac{B}{x+1}$$
$$5x - 7 \equiv A(x+1) + B(x-3) = (A+B)x + (A-3B).$$

Hence

$$A + B = 5, A - 3B = -7.$$

So $A = 2, B = 3$.

$$\begin{aligned}\int \frac{5x-7}{x^2-2x-3}dx &= \int \frac{2}{x-3} + \frac{3}{x+1}dx \\ &= 2\ln|x-3| + 3\ln|x+1| + C.\end{aligned}$$

■

Exercise 9.5.1. $\displaystyle\int \frac{x-2}{2x^2-5x+3}\,dx = -\frac{1}{2}\ln|2x-3| + \ln|x-1| + C$

case (ii) : $\Delta = 0, ax^2 + bx + c = a(x - x_1)^2$

Express

$$\frac{ax+b}{ax^2+bx+c} = \frac{A}{x-x_1} + \frac{B}{(x-x_1)^2}.$$

Example 9.5.3. Evaluate

$$\int \frac{2x-1}{(x-2)^2}dx.$$

Solution. Suppose

$$\frac{2x-1}{(x-2)^2} \equiv \frac{A}{x-2} + \frac{B}{(x-2)^2}.$$

Hence

$$2x-1 \equiv A(x-2) + B = Ax + (B - 2A).$$

Thus

$$2 = A, -1 = B - 2A$$

$$A = 2, B = 3.$$

$$\begin{aligned}\int \frac{2x-1}{(x-2)^2} &= \int \frac{2}{x-2} + \frac{3}{(x-2)^2}dx \\ &= 2\ln|x-2| - \frac{3}{x-2} + C.\end{aligned}$$

■

Exercise 9.5.2. $\displaystyle\int \frac{4x+2}{(2x-1)^2}\,dx = \ln|2x-1| - \frac{2}{2x-1} + C$

Remark.

1. the subcase (iii) $\Delta < 0$ involves trigonometric function, and it is not required in this course!

2. For other cases $\deg q(x) > 2$, the idea is the same: apply partial fraction decomposition, which is not required also!

Example 9.5.4. Evaluate

$$\int \frac{x^5}{x^2-1}dx.$$

Solution.

$$\begin{array}{r|l} & x^3 + x. \\ x^2 - 1 & x^5 \\ & -x^5 + x^3 \\ & x^3 \\ & -x^3 + x \\ & x \end{array}$$

$$x^5 = (x^2 - 1)(x^3 + x) + x.$$

$$\frac{x}{x^2 - 1} = \frac{x}{(x-1)(x+1)} = \frac{1}{2(x-1)} + \frac{1}{2(x+1)}.$$

Thus

$$\begin{aligned} \int \frac{x^5}{x^2 - 1} &= \int x^3 + x + \frac{1}{2(x-1)} + \frac{1}{2(x+1)} dx \\ &= \frac{x^4}{4} + \frac{x^2}{2} + \frac{1}{2} \ln|x-1| + \frac{1}{2} \ln|x+1| + C \end{aligned}$$

■

Exercise 9.5.3. $\displaystyle\int \frac{4x^2 - 7x + 5}{x^2 - 2x + 1}\, dx = 4x + \ln|x-1| - \frac{2}{x-1} + C$