MATH1520 University Mathematics for Applications	Fall 2021

Chapter 6: Application of Derivatives I

Learning Objectives:
(1) Apply L'Hôpital's rule to find limits of indeterminate forms.
(2) Discuss increasing and decreasing functions.
(3) Define critical points and relative/absolute extrema of real functions of 1 variable.
(4) Use the first derivative test to study relative/absolute extrema of functions.

6.1 Limits of indeterminate forms and L'Hôpital's rule

Recall the Remark in the end of Section 2.4 regarding exceptional cases of limits, which can not be computed using the algebraic rules of limits in Proposition 2, but the limits might still exist. Limits of this type are said to be of indeterminate forms.

6.1.1 Limits of indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$

Consider $\lim_{x\to a} \frac{f(x)}{g(x)}$,

1. if $\lim_{x\to a} f(x) = A$, $\lim_{x\to b} g(x) = B \neq 0$, $A, B \in \mathbb{R}$, then by the quotient rule,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{A}{B}.$$

2. if $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad (\pm\infty)$, then the quotient rule is not applicable. Limits of this type are said to be of indeterminate form type $\frac{0}{0}$ or type $\frac{\infty}{\infty}$

   For example,

$$\lim_{x\to 1} \frac{x^2-1}{x^3-1}, \quad \left(\text{type } \frac{0}{0}\right)$$

$$\lim_{x\to+\infty} \frac{x+1}{2x+3}, \quad \lim_{x\to+\infty} \frac{-x+1}{2x^3}, \quad \left(\text{type } \frac{\infty}{\infty}\right).$$

Theorem 6.1.1 (L'Hôpital's rule for limits of types $\frac{0}{0}, \frac{\infty}{\infty}$).

Let $f(x)$, $g(x)$ be differentiable and suppose that $g'(x) \neq 0$ near the point a.

If

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad \text{or} \quad \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty,$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Remark. (a) An intuitive explanation: When $f(a) \approx 0 \approx g(a)$,

$$\frac{f(x)}{g(x)} \approx \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}.$$

(b) The statement of the theorem still holds if "$x \to a$" is replaced by "$x \to \pm\infty$" or "$x \to a^{\pm}$". It also holds if $\lim_{x\to a} f(x) = \pm\infty$ $\lim_{x\to a} g(x) = \mp\infty$. (Use $\lim_{x\to a} \frac{f(x)}{g(x)} = -\lim_{x\to a} \frac{-f(x)}{g(x)}$ and apply the theorem to $\lim_{x\to a} \frac{-f(x)}{g(x)}$.)

Example 6.1.1. Limits of type $\frac{0}{0}$

1.

$$\begin{aligned} &\lim_{x\to 1} \frac{x^2-1}{x^3-1} && \text{(check condition 1: } \tfrac{0}{0}\text{)} \\ =&\lim_{x\to 1} \frac{2x}{3x^2} && \text{(check condition 2: this limit is } \tfrac{2}{3}\text{)} \\ =&\frac{2}{3}. \end{aligned}$$

Remark. Alternatively, use the "canceling common factors" trick in the previous chapters.

2.

$$\begin{aligned} &\lim_{x\to 1} \frac{e^x - e}{\sqrt{x}-1} && \text{(the limit is of type } \tfrac{0}{0}\text{)} \\ =&\lim_{x\to 1} \frac{e^x}{\frac{1}{2}x^{-\frac{1}{2}}} \\ =&2e. \end{aligned}$$

3.

$$
\begin{aligned}
&\lim_{x\to 0^+}\frac{\ln(1+x)}{x^2} \qquad (\text{type } \frac{0}{0})\\
=&\lim_{x\to 0^+}\frac{\frac{1}{1+x}}{2x}\\
=&+\infty.
\end{aligned}
$$

Example 6.1.2. Limits of type $\frac{\infty}{\infty}$

1.

$$
\begin{aligned}
&\lim_{x\to+\infty}\frac{-x+1}{2x+3} \qquad (\text{type } \frac{\infty}{\infty})\\
=&\lim_{x\to+\infty}\frac{-1}{2}\\
=&-\frac{1}{2}.
\end{aligned}
$$

Remark. The same result can be obtained by dividing both the numerator and the denominator by x.

2.

$$
\begin{aligned}
&\lim_{x\to+\infty}\frac{\ln x}{x^n}, n\in\mathbb{N} \qquad (\text{type } \frac{\infty}{\infty})\\
=&\lim_{x\to+\infty}\frac{\frac{1}{x}}{nx^{n-1}}\\
=&\lim_{x\to+\infty}\frac{1}{nx^n}\\
=&0.
\end{aligned}
$$

Remark.

1. L'Hôpital's rule can NOT be applied for determinate form.
   For example, $\lim_{x\to 1}\frac{x+1}{x+2}=\frac{2}{3}$, but $\lim_{x\to 1}\frac{(x+1)'}{(x+2)'}=\frac{1}{1}=1$.
2. If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ is still $\frac{0}{0}$, $\frac{\infty}{\infty}$, then repeat L'Hôpital's rule.
3. L'Hôpital's rule can be used to justify the previous assertion that as $x\to\infty$, higher degree polynomials "grows faster" than lower degree polynomials; exponential functions grow faster than any polynomials; log functions grow slower than any polynomials.

Exercise 6.1.1.

1. $\lim_{x\to 1} \frac{x-1}{\ln x} = 1$
2. $\lim_{x\to +\infty} \frac{x^n}{e^x} = 0$

Example 6.1.3. (Applying L'Hôpital's rule twice.)

$$\begin{aligned} &\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x^2} \qquad (\text{type } \frac{0}{0}) \\ =&\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{2x} \qquad (\text{ still of type } \frac{0}{0}) \\ =&\lim_{x\to 0} \frac{e^x - e^{-x}}{2} \\ =&0 \end{aligned}$$

6.1.2 Other Indeterminate Forms: $0 \cdot \infty$, $\infty - \infty$, 0^0, 1^∞, ∞^0

All these forms can be converted to forms of types $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 6.1.4. Type $0 \cdot \infty$

$$\begin{aligned} &\lim_{x\to 0^+} (x \ln x) \qquad (0 \cdot \infty) \\ =&\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \qquad (\frac{\infty}{\infty}) \\ =&\lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\ =&\lim_{x\to 0^+} (-x) \\ =&0. \end{aligned}$$

Example 6.1.5. Type $\infty - \infty$

$$\begin{aligned}
&\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) && (\infty-\infty)\\
=&\lim_{x\to 0^+}\frac{e^x-1-x}{x(e^x-1)} && (\frac{0}{0})\\
=&\lim_{x\to 0^+}\frac{e^x-1}{e^x-1+xe^x} && (\text{ still } \frac{0}{0})\\
=&\lim_{x\to 0^+}\frac{e^x}{e^x+e^x+xe^x}\\
=&\frac{1}{2}.
\end{aligned}$$

Example 6.1.6. Types 1^∞, ∞^0, 0^0

Trick: $\boxed{f^g = e^{\ln f^g} = e^{g\ln f}}$

1.

$$\begin{aligned}
&\lim_{x\to+\infty} x^{\frac{1}{x}} && (\infty^0)\\
=&\lim_{x\to+\infty} e^{\ln(x^{\frac{1}{x}})}\\
=&\lim_{x\to+\infty} e^{\frac{1}{x}\ln x}\\
=&e^{\lim\limits_{x\to+\infty}\frac{1}{x}\ln x},
\end{aligned}$$

$$\begin{aligned}
&\lim_{x\to+\infty}\frac{1}{x}\ln x && (0\cdot\infty)\\
=&\lim_{x\to+\infty}\frac{\ln x}{x} && (\frac{\infty}{\infty})\\
=&\lim_{x\to+\infty}\frac{\frac{1}{x}}{1}\\
=&0.
\end{aligned}$$

So,

$$\lim_{x\to+\infty} x^{\frac{1}{x}} = e^0 = 1.$$

2.

$$\begin{aligned}
&\lim_{x\to 1^+} x^{\frac{1}{1-x}} && (1^\infty)\\
=&\lim_{x\to 1^+} e^{\frac{1}{1-x}\ln x}\\
=&e^{\lim\limits_{x\to 1^+}\frac{\ln x}{1-x}},
\end{aligned}$$

$$\begin{aligned}&\lim_{x\to 1^+}\frac{\ln x}{1-x} \qquad (\frac{0}{0})\\ =&\lim_{x\to 1^+}\frac{\frac{1}{x}}{-1}\\ =&-1.\end{aligned}$$

So,

$$\lim_{x\to 1^+} x^{\frac{1}{1-x}} = e^{-1}.$$

3.

$$\begin{aligned}&\lim_{x\to 0^+} x^x \qquad (0^0)\\ =&\lim_{x\to 0^+} e^{x\ln x}\\ =&e^{\lim_{x\to 0^+} x\ln x},\end{aligned}$$

$$\begin{aligned}&\lim_{x\to 0^+} x\ln x \qquad (0\cdot\infty)\\ =&\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} \qquad (\frac{\infty}{\infty})\\ =&\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}\\ =&\lim_{x\to 0^+}(-x)\\ =&0.\end{aligned}$$

So,

$$\lim_{x\to 0^+} x^x = e^0 = 1.$$

6.2 Monotonicity of Functions and the First Derivative Test

6.2.1 Monotonicity: Increasing/Decreasing Functions

Definition 6.2.1. Let $f(x)$ be a function defined on (a,b). Then

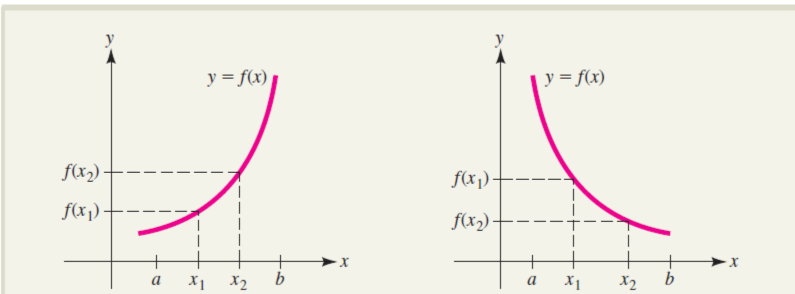
1. $f(x)$ is **increasing** (or *positively monotone*) on the interval if $f(x_2)\geq f(x_1)$ whenever $x_2 > x_1$.
2. $f(x)$ is **strictly increasing** (or *strictly positive monotone*) on the interval if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.

3. $f(x)$ is **decreasing** (or *negatively monotone*) on the interval if $f(x_2) \leq f(x_1)$ whenever $x_2 < x_1$.

4. $f(x)$ is **strictly decreasing** (or *strictly negative monotone*) on the interval if $f(x_2) < f(x)$ whenever $x_2 > x_1$.

5. $f(x)$ is *(strictly) monotone* if $f(x)$ is either (strictly) positively monotone or (strictly) negatively monotone.

Caveat! The preceding definition is the mathematicians' definition of increasing/decreasing functions. However, some calculus texts define increasing/decreasing functions differently, e.g. [Hoffmann et al.], where "increasing/descreasing functions" refer to the "strictly increasing/descreasing functions" defined above. Similarly, some text refers to what we called "strictly monotone/monotone" above as "monotone/weakly monotone".



(a) $f(x)$ is increasing on $a < x < b$ (b) $f(x)$ is decreasing on $a < x < b$

Theorem 6.2.1. *Let f be a differentiable function on (a, b).*

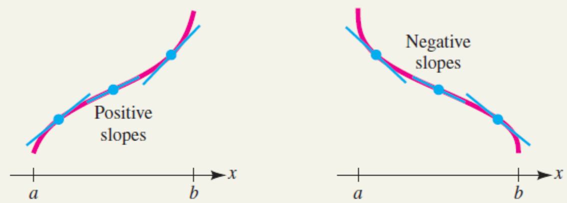
1. *If $f'(x) \geq 0$ for all $x \in (a, b)$, then $f(x)$ is an increasing function.*
2. *If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is a strictly increasing function on (a, b).*
3. *If $f'(x) \leq 0$ for all $x \in (a, b)$, then $f(x)$ is a decreasing function.*
4. *If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is a strictly decreasing function on (a, b).*

Example 6.2.1. Show that $f(x) = e^x - x - 1$ is a strictly increasing function on $(0, \infty)$.

Solution. $f'(x) = e^x - 1 > 1 - 1 = 0$. So $f(x)$ is a strictly increasing function. ■

Remark. Because $f(x)$ is a strictly increasing function, $f(x) > f(0) = 0$ for $x > 0$, i.e.

$$e^x > 1 + x, \text{ for } x > 0.$$



(a) $f'(x) > 0$ on $a < x < b$, so $f(x)$ is increasing. (b) $f'(x) < 0$ on $a < x < b$, so $f(x)$ is decreasing.

Procedure to determine intervals of increase/decrease of f

1. Find all c such that $f'(c) = 0$ or $f'(c)$ is undefined. Divide the line into several intervals.
2. For each intervals (a, b) obtained in the previous step.
   (a) If $f'(x) > 0$, $f(x)$ is a strictly increasing function (↑) on (a, b).
   (b) If $f'(x) < 0$, $f(x)$ is a decreasing function (↓) on (a, b).

Example 6.2.2. Find the intervals in which the function

$$f(x) = 2x^3 + 3x^2 - 12x - 7$$

is strictly increasing/strictly decreasing.

Solution.

$$f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1) = 0 \quad \Rightarrow x = -2, 1.$$

So we have 3 intervals: $(-\infty, -2)$, $(-2, 1)$, $(1, \infty)$.

In $(-\infty, -1)$, $x + 1 < 0, x - 1 < 0$, so $f'(x) > 0$.
In $(-1, 1)$, $x + 1 > 0, x - 1 < 0$, so $f'(x) < 0$.
In $(1, +\infty)$, $x + 1 > 0, x - 1 > 0$, so $f'(x) > 0$.

x	$(-\infty, -2)$	-2	$(-2, 1)$	1	$(1, +\infty)$
$f'(x)$	$+$	0	$-$	0	$+$
monotonicity	↑		↓		↑

■

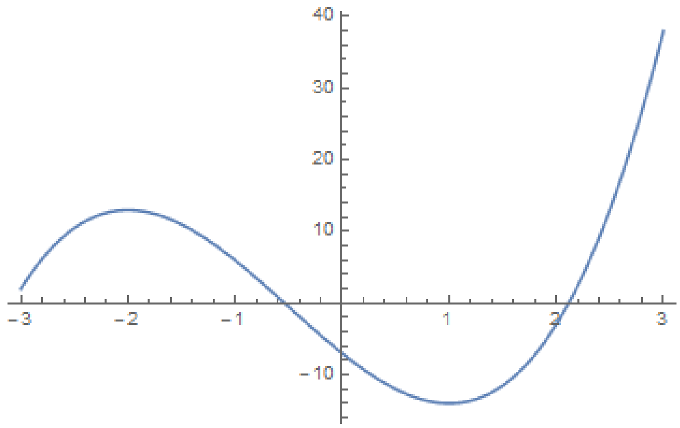


Figure 6.1: $y = 2x^3 + 3x^2 - 12x - 7$

Exercise 6.2.1. Find the intervals of strict increase and strict decrease of the function

$$f(x) = x^7 - 2x^5 + x^3.$$

Solution.

$$f'(x) = 7x^6 - 10x^4 + 3x^2 = x^2(7x^2-3)(x^2-1) = 0 \quad \Rightarrow \quad x = 0, \pm 1 \text{ and } \pm\sqrt{\frac{3}{7}} \approx \pm 0.654654.$$

x	$(-\infty,-1)$	$(-1,-\sqrt{\frac{3}{7}})$	$(-\sqrt{\frac{3}{7}},0)$	$(0,\sqrt{\frac{3}{7}})$	$(\sqrt{\frac{3}{7}},1)$	$(1,+\infty)$
$f'(x)$	$+$	$-$	$+$	$+$	$-$	$+$
monotonicity	$\uparrow$	$\downarrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$

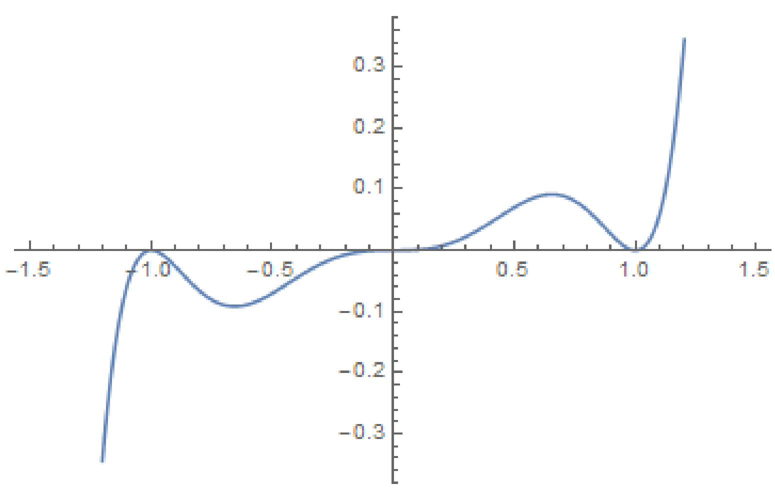


Figure 6.2: $y = x^7 - 2x^5 + x^3$

■

Definition 6.2.2. Let $f(x)$ be a real-valued function defined on (a, b). A number $c \in (a, b)$ is called a critical point of f if $f'(c) = 0$ or $f'(c)$ does not exist.
The corresponding value $f(c)$ is called a critical value for $f(x)$.

Remark. The notion of critical points applies to more general functions, e.g. real functions of several variables, complex functions etc. A critical point always lies in the domain of the function. In the special case of real-valued functions of a single real variable, a critical point is a real number; therefore it is also called a *critical number*. Let $f(x)$ be a real-valued function of a single real variable, and $c \in \mathbb{R}$ be a critical point of f. Let $C \subset \mathbb{R}^2$ be the graph of f in the $x - y$ plane. The point $(c, f(c)) \in C$ is a critical point of the function $\pi_y : C \to \mathbb{R}$ given by $(x, y) \mapsto y$.

Example 6.2.3.

$$f(x) = |x|.$$

We have proved

$$f'(x) = \begin{cases} -1, & x < 0, \\ \text{does not exist}, & x = 0, \\ 1, & x > 0. \end{cases}$$

$\Rightarrow$ critical number: $x = 0$; corresponding critical value: 0

x	$(-\infty, 0)$	0	$(0, +\infty)$
$f'(x)$	$-$	0	$+$
monotonicity	$\downarrow$		$\uparrow$

Example 6.2.4. $f(x) = x^4 - 4x^3$. Find all critical points and increasing & decreasing intervals.

Solution.

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3) = 0 \quad \Rightarrow \quad x = 0, 3.$$

critical points: $x = 0, 3$
corresponding critical values: $f(0) = 0, f(3) = -27$

x	$(-\infty, 0)$	0	$(0, 3)$	3	$(3, +\infty)$
$f'(x)$	$-$	0	$-$	0	$+$
monotonicity	$\downarrow$		$\downarrow$		$\uparrow$

■

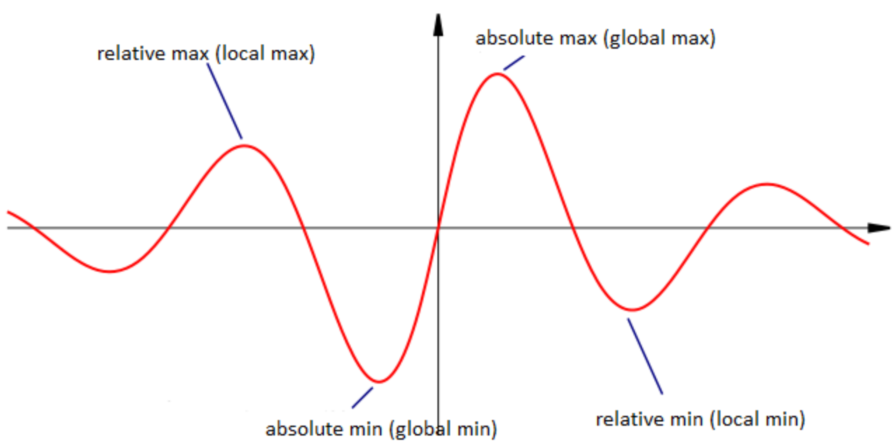
6.2.2 Maxima & Minima of Functions

Definition 6.2.3. Let $f(x)$ be a real-valued function with domain I. We say

1. $f(x)$ has a **relative maximum** (or **local maximum**) at $x = c$ if $f(c) \geq f(x)$ for all $x \in I$ near c.
2. $f(x)$ has a **global maximum** (or **absolute maximum**) at $x = c$ if $f(c) \geq f(x)$ for all $x \in I$.

Similar definition for relative/global minimum.

Both maximum and minimum are called an extremum.



Remark. Global extremum $\Rightarrow$ Local extremum
But Global extremum $\not\Leftarrow$ Local extremum

Remark. There is some confusion in the literature regarding whether a (local or global) maximum/minimum of a function refers to an element in the domain or its corresponding value (in the range). For most literature, *the* (absolute) maximum of a real function $f(x)$ refers to the value: $M \in \mathbb{R}$ *is said to be the (absolute) maximum if there exists an element* c *in the domain* D *of* f *such that* $f(x) \leq f(c)\ \forall x \in D$. To be clear, say that M is an (absolute) maximum *value* of f; and f *attains* its (absolute) maximum *at* c. Say e.g. f has local maxima *at* $x_1, x_2, \ldots \in D$, with corresponding values $f(x_1), f(x_2), \ldots$. Similarly for the notions of (absolute/local) minimum.

Remark. Absolute maxima/minima may not exist. Consider the e.g. the function $f : (0, 1] \to \mathbb{R}$ given by $f(x) = x$. This f has an absolute maximum but has no absolute minimum. A general notion is *supremum/infimum*. In the above example, the supremum of f is 1 and its infimum is 0.

Question I: How to find relative extrema?

Theorem 6.2.2 (First Derivative Test: Relative Extrema).

Let $f(x)$ be a continuous function which is differentiable where $x \neq c$. Then

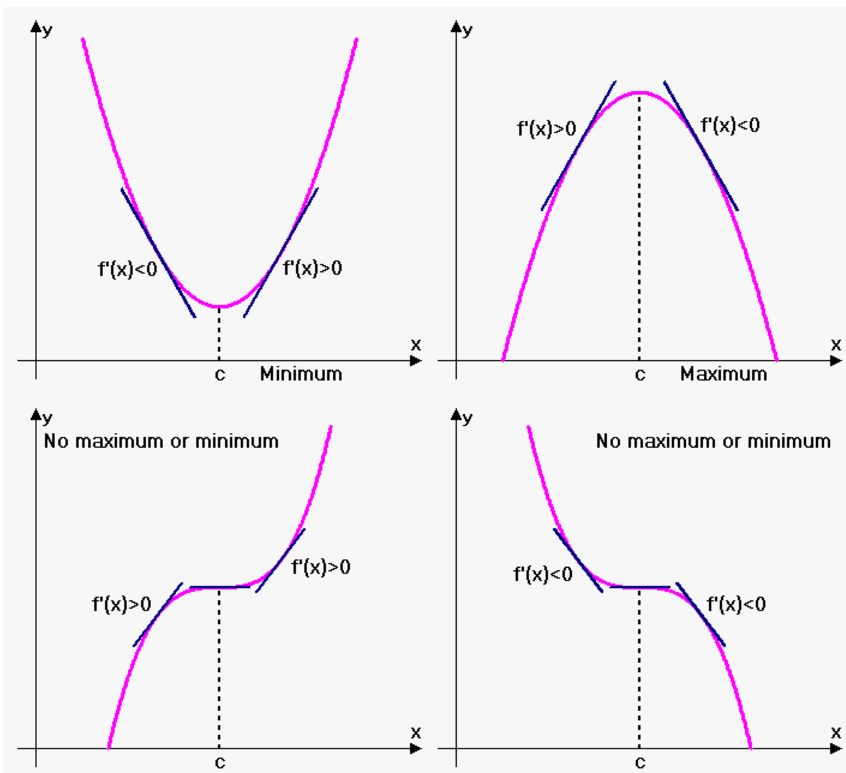
1. *$f(x)$ attains a relative maximum at $x = c$ if near the point c,*

$$f'(x) > 0 \quad \text{for } x < c; \quad f'(x) < 0 \quad \text{for } x > c.$$

2. *$f(x)$ attains a relative minimum at $x = c$ if near the point c,*

$$f'(x) < 0 \quad \text{for } x < c; \quad f'(x) > 0 \quad \text{for } x > c.$$

3. *$f(x)$ attains no relative extremum at $x = c$ if near the point c, $f'(x)$ has the same sign on two sides of c.*



Property	Sign of $f'(x)$ to the left of c	Sign of $f'(x)$ to the right of c
Relative maximum	$+$	$-$
Relative minimum	$-$	$+$
Not a relative extremum	$+$	$+$
Not a relative extremum	$-$	$-$

Theorem 6.2.3. *Let $c \in (a, b)$ and let f be a continuous function on (a, b) such that f' exists and is continuous on $(a, b)\backslash\{c\}$. Then f attains a relative extremum at $x = c \quad \Rightarrow \quad c$ is a critical number, i.e. $f'(c) = 0$ or $f'(c)$ does not exist.*

Remark. f attains a relative extremum at $x = c \quad \not\Leftarrow \quad c$ is a critical number.
For example, $f(x) = x^3, f'(x) = 3x^2$, so $x = 0$ is a critical number. But $f'(x) > 0$ on two sides of $x = 0$, so f does not have a relative extremum at 0.

Example 6.2.5. Let

$$f(x) = 2x^3 + 3x^2 - 12x - 7.$$

Find all its relative maxima and relative minima.

Solution. Refer to the answer of Example 6.2.2, $f'(x) = 6x^2 + 6x - 12$. The critical numbers are solutions of $f'(x) = 0$, i.e $x = -2$ and $x = 1$.

x	$(-\infty, -2)$	-2	$(-2, 1)$	1	$(1, +\infty)$
$f'(x)$	$+$	0	$-$	0	$+$

(point where a relative maximum occurs, corresponding value): $(-2, f(-2)) = (-2, 13)$
(point where a relative minimum occurs, corresponding value): $(1, f(1)) = (1, 14)$

■

Example 6.2.6.

1. For Example 6.2.3 $f(x) = |x|$.
   One critical number: $x = 0$, One relative minimum at 0, with corresponding value 0.
2. For example 6.2.4 $f(x) = x^4 - 4x^3$.
   critical numbers: $x = 0, 3$, one relative minimum at 3, with corresponding value -27.

Exercise 6.2.2. Let

$$f(x) = x^7 - 2x^5 + x^3.$$

(see Exercise 6.2.1) Find all relative maxima and relative minima of f.

Answer:

$$\text{(point where a relative maximum occurs, corresponding value)}:$$
$$(-1, f(-1)) = (-1, 0); (\sqrt{\tfrac{3}{7}}, f(\sqrt{\tfrac{3}{7}}) \approx (0.655, 0.092)$$
$$\text{(point where a relative minimum occurs, corresponding value)}:$$
$$(-\sqrt{\tfrac{3}{7}}, f(-\sqrt{\tfrac{3}{7}})) \approx (-0.655, -0.092); (1, f(1)) = (1, 0).$$

Note that f has no relative extremum at 0.

Question II: How to find absolute Max/Min?

Theorem 6.2.4. *Suppose $f : [a, b] \to \mathbf{R}$ is a continuous function, then the absolute maximum point and absolute minimum point exist for the graph of f (Theorem 3.2.2 Extreme Value Theorem).*

Remark. Note that the preceding theorem applies only when the domain of f is a *closed finite* interval!

Procedures to find absolute max/min of continuous function f on $[a, b]$

1. Find all the critical numbers $c_1, c_2, \ldots,$ in (a, b).
2. Compute the values $f(a), f(b), f(c_1), f(c_2), \ldots,$
   The maximum value corresponds to the absolute max.
   The minimum value corresponds to the absolute min.

Example 6.2.7. Find the absolute maximum and absolute minimum of $f(x) = x^5 - 80x$ on $[-3, 4]$.

Solution. Since $f(x)$ is continuous on $[-3, 4]$, the absolute max/min can be reached by extreme value theorem.

$$f'(x) = 5x^4 - 80 = 0 \quad \Rightarrow \quad x = -2, 2.$$

Compute

$$f(-2) = 128, \quad f(2) = -128,$$
$$f(-3) = -3, \quad f(4) = 704.$$

The absolute minimum is -128, attained at $x = 2$; the absolute maximum is 704, attained at $x = 4$.

■

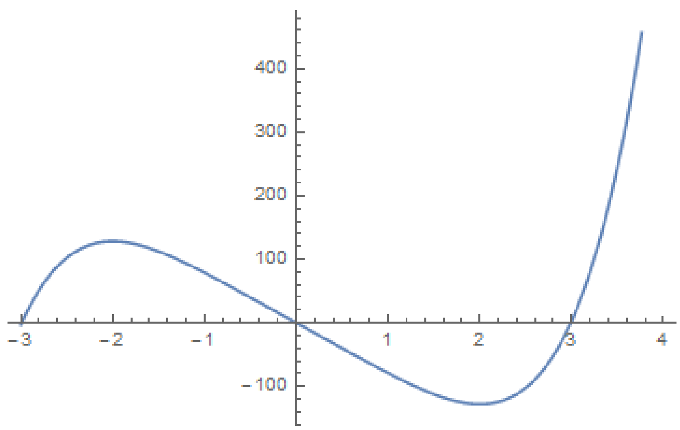


Figure 6.3: $y = x^5 - 80x$ over $[-3, 4]$