MATH1520 University Mathematics for Applications **Fall 2021**

Chapter 5: Differentiation II

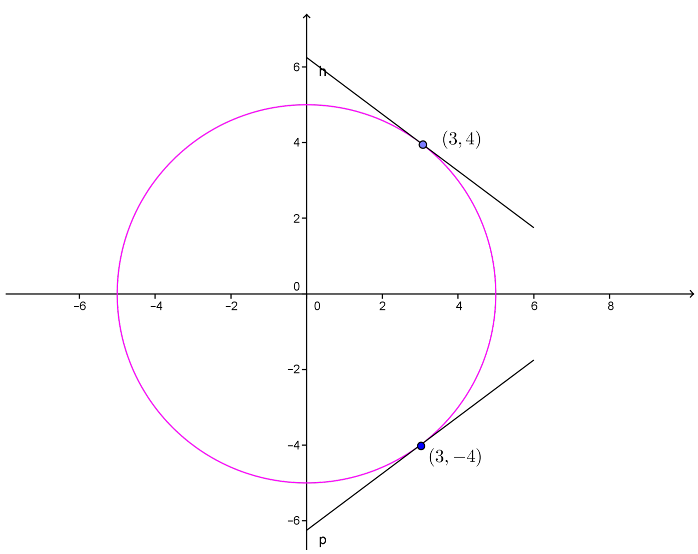
Learning Objectives:
(1) Use implicit differentiation to find slope.
(2) Discuss inverse function and its derivatives.
(3) Study the higher order derivative.

5.1 Differentiating Implicit Functions and Inverse Functions

5.1.1 Implicit functions

Example 5.1.1. Consider the circle on the $x - y$ plane defined by $x^2 + y^2 = 25$. Find the equation of the tangent line to the circle at $(3, 4)$.

Solution. **Method 1. Express y in terms of x explicitly.**



$$x^2 + y^2 = 25 \quad \Rightarrow y = \pm\sqrt{25 - x^2},$$

Restrict to a small neighbourhood of the point $(3, 4)$ on the curve, $y > 0$ can be uniquely given by $y = \sqrt{25 - x^2}$.

So,

$$y' = -\frac{x}{\sqrt{25 - x^2}}$$

when $x = 3$, $y' = -\frac{3}{4}$. The equation of the tangent line to the curve at $(3, 4)$ is

$$\begin{aligned} y - 4 &= -\frac{3}{4}(x - 3), \\ y &= -\frac{3}{4}x + \frac{25}{4}. \end{aligned}$$

Method 2. Implicit differentiation.

Regard y as a function $y(x)$ without explicit formula. Differentiate both sides of $x^2+y^2 = 25$ with respect to x, and then solve algebraically for $\frac{dy}{dx}$.

$$\begin{aligned} 2x + \frac{d}{dx}(y^2) &= 0 \\ 2x + 2y\frac{dy}{dx} &= 0 \quad \text{(chain rule)} \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

So,

$$\left.\frac{dy}{dx}\right|_{(3,4)} = -\frac{3}{4}.$$

Then, find the tangent line in the same way as with Method 1.

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Remark. Method 2 is referred to as implicit differentiation, which is very useful to compute derivatives of functions not defined by explicit formulae.

Example 5.1.2. Let $y = f(x)$ be a differentiable function of x that satisfies the equation $x^2y + y^2 = x^3$. Find the derivative $\dfrac{dy}{dx}$ as a function of both x and y.

Solution. You are going to differentiate both sides of the given equation with respect to x. So that you will not forget that y is actually a function of x, temporarily use the alternative notation $f(x)$ for y, and begin by rewriting the equation as

$$x^2f(x) + (f(x))^2 = x^3.$$

Now differentiate both sides of this equation term by term with respect to x:

$$\begin{aligned} \frac{d}{dx}[x^2 f(x) + (f(x))^2] &= \frac{d}{dx}[x^3] \\ \rightsquigarrow \left[x^2\frac{df}{dx} + f(x)\frac{d}{dx}(x^2)\right] + 2f(x)\frac{df}{dx} &= 3x^2. \end{aligned} \tag{5.1}$$

Thus, we have

$$\begin{aligned} x^2\frac{df}{dx} + f(x)(2x) + 2f(x)\frac{df}{dx} &= 3x^2 \\ \rightsquigarrow [x^2 + 2f(x)]\frac{df}{dx} &= 3x^2 - 2xf(x) \\ \rightsquigarrow \frac{dy}{dx} &= \frac{3x^2 - 2xf(x)}{x^2 + 2f(x)}. \end{aligned} \tag{5.2}$$

Finally, replace $f(x)$ by y to get

$$\frac{dy}{dx} = \frac{3x^2 - 2xy}{x^2 + 2y}.$$

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Remark. By default, $\frac{dy}{dx}$ is regarded as a function of x, and we want an expression for $\frac{dy}{dx}$ in terms of x only. However, sometimes it is difficult to express y in terms of x explicitly. In this case it'll be specified in the test or homework question that it is ok to leave the answer for y' as a function of both x and y. Or, sometimes finding the value for y' is only an intermediate step in solving the problem. If the values of x and y are known, one may directly plug in these values to the expression of y' in x and y, without going through an explicit formula for y' in x.

Summary: Carrying out Implicit Differentiation

Suppose an equation defines y implicitly as a differentiable function of x. To find $\frac{dy}{dx}$:

1. Differentiate both sides of the equation with respect to x. Remember that y is really a function of x, and use the chain rule when differentiating terms containing y.
2. Solve the differentiated equation algebraically for $\frac{dy}{dx}$ in terms of x and y.

Example 5.1.3. Consider the curve defined by

$$x^3 + y^3 = 9xy.$$

1. Compute $\frac{dy}{dx}$. (It is ok to leave the answer as a function of both x and y.)
2. Find the slope of the tangent line to the curve at $(4, 2)$.

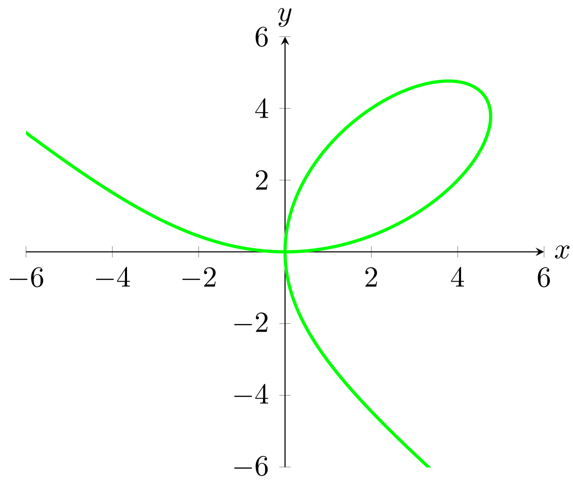

Figure 5.1: A plot of $x^3 + y^3 = 9xy$. While this is not a function of y in terms of x, the equation still defines a relation between x and y.

Solution. Starting with

$$x^3 + y^3 = 9xy,$$

we apply the differential operator $\frac{d}{dx}$ to both sides of the equation to obtain

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}9xy.$$

Applying the sum rule, we see that

$$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = \frac{d}{dx}9xy.$$

Let's examine each of the terms above in turn. To begin,

$$\frac{d}{dx}x^3 = 3x^2.$$

On the other hand, $\frac{d}{dx}y^3$ is treated somewhat differently. Here, viewing $y = y(x)$ as an implicit function of x, we have by the chain rule that

$$\begin{aligned}\frac{d}{dx}y^3 &= \frac{d}{dx}(y(x))^3 \\ &= 3(y(x))^2 \cdot y'(x) \\ &= 3y^2\frac{dy}{dx}.\end{aligned}$$

Consider the final term $\dfrac{d}{dx}(9xy)$. Regarding $y = y(x)$ again as an implicit function, we have:

$$\begin{aligned} \frac{d}{dx}(9xy) &= 9\frac{d}{dx}\big(x \cdot y(x)\big) \\ &= 9\left(x \cdot y'(x) + y(x)\right) \\ &= 9x\frac{dy}{dx} + 9y. \end{aligned}$$

Putting all the above together, we get:

$$3x^2 + 3y^2\frac{dy}{dx} = 9x\frac{dy}{dx} + 9y.$$

Now we solve the preceding equation for $\dfrac{dy}{dx}$. Write

$$\begin{aligned} & 3x^2 + 3y^2\frac{dy}{dx} = 9x\frac{dy}{dx} + 9y \\ \iff\quad & 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} = 9y - 3x^2 \\ \iff\quad & \frac{dy}{dx}\left(3y^2 - 9x\right) = 9y - 3x^2 \\ \iff\quad & \frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}. \end{aligned}$$

For the second part of the problem, we simply plug in $x = 4$ and $y = 2$ to the last formula above to conclude that the slope of the tangent line to the curve at $(4, 2)$ is $\dfrac{5}{4}$. See Figure 5.2. ■

Example 5.1.4. Let L be the curve in the $x - y$ plane defined by $x^2 + y^2 + e^{xy} = 2$. Use L to implicitly define a function $y = y(x)$. Find $y'(x)$ at $x = 1$ and the tangent line to the curve L at $(1, 0)$.

Solution. (Note: In this case, there is no good explicit formula for the function $y(x)$.) Differentiate the equation $x^2 + y^2 + e^{xy} = 2$ on both sides with respect to x. We get:

$$\begin{aligned} 2x + 2yy' + e^{xy}(y + xy') = 0, \\ \rightsquigarrow y' = -\frac{2x + e^{xy}y}{2y + e^{xy}x}. \end{aligned}$$

So, $y(1) = 0$ and $y'|_{x=1)} = -2$.

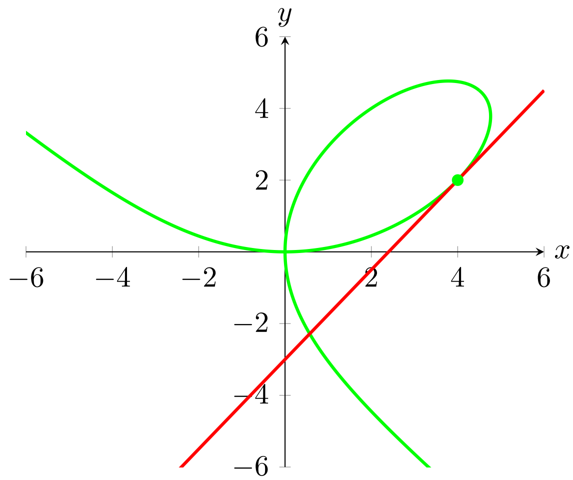

Figure 5.2: A plot of $x^3 + y^3 = 9xy$ along with the tangent line at $(4, 2)$.

Thus, the equation of the tangent line to L at $(x, y) = (1, 0)$ is:

$$y - 0 = -2(x - 1), \quad \text{or}$$
$$y = -2x + 2.$$

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5.1.2 Differentiating Inverse Functions

Definition 5.1.1. Consider a function $f : A \to B$, where A is the domain, and B is the codomain.

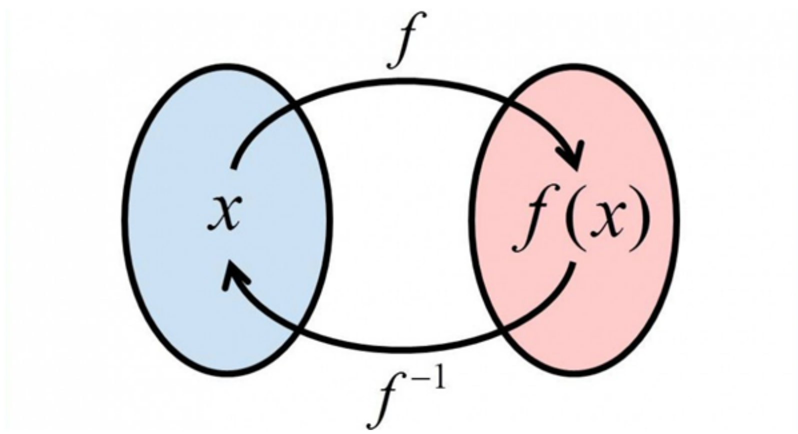
The function f is said to be *injective* if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ for any $x_1, x_2 \in A$. The function f is said to be *surjective* or *onto* if $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$. (In this case, the codomain B of f agrees with the range of f.) The function f is said to be *bijective* or *one to one* if it is both injective and surjective.

If f is one-to-one, then the *inverse function*, denoted $f^{-1} : B \to A$, is defined by

$$x = f^{-1}(y) \quad \text{if } y = f(x).$$

Remark.

1. Only a one-to-one function can have an inverse.



2. The domains and codomains(=ranges) of f and f^{-1} are interchanged.
3. $f^{-1}(x)$ is **not** $\dfrac{1}{f(x)}$.
4.

$$\begin{aligned} (f^{-1} \circ f)(x) &= x, && \text{for all } x \text{ in the domain of } f \\ (f \circ f^{-1})(y) &= y, && \text{for all } y \text{ in the domain of } f^{-1} \text{ (or range of } f) \end{aligned}$$

Example 5.1.5.

1.

$$\begin{cases} y = e^x, \\ x = \ln y. \end{cases} \qquad x \in \mathbb{R}, y > 0$$

   are inverse functions of each other.

2.

$$\begin{cases} y = x^2, \\ x = \sqrt{y}. \end{cases} \qquad x > 0, y > 0$$

   are inverse functions of each other.

3. $y = x^2$, $x \in \mathbb{R}, y \geq 0$ does not have inverse function because it is not one-to-one.

Question: What is the relation between derivatives of inverse functions?

Suppose $y = f(x)$ has an inverse function, then

$$x = f^{-1}(f(x)).$$

Differentiate both sides with respect to x to get:

$$1 = (f^{-1})'(y) \cdot f'(x)$$

$$\iff \quad \boxed{(f^{-1})'(y) = \frac{1}{f'(x)},}$$

or equivalently,

$$\boxed{\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.}$$

Example 5.1.6. Use the identity $\dfrac{d}{dx}e^x = e^x$ to show that

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Solution. Let $y = f(x) = \ln x$. Then its inverse function is $x = e^y$.

$$\frac{dy}{dx} = \frac{d}{dx}\ln x = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y}.$$

Express the right hand side in terms of x, we have

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Or, using implicit differentiation: Differentiate the equation $x = e^y$ on both sides with respect to x. We get:

$$1 = \frac{d}{dx}(e^y) = e^y \cdot \frac{dy}{dx} \qquad \text{(the chain rule)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\ln x = \frac{1}{e^y} = \frac{1}{x}.$$

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Example 5.1.7. Show that

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Solution. Let $y = \sqrt{x}$, then $x = y^2$. We have:

$$\frac{d\sqrt{x}}{dx} = \frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{2y}.$$

Expressing the right hand side in terms of x, we have

$$\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}}.$$

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Example 5.1.8. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^3 + 4x$.

1. Find $\frac{d}{dx} f^{-1}(x)$ without writing down an explicit formula for $f^{-1}(x)$.
2. Find $\left.\frac{d}{dx} f^{-1}(x)\right|_{x=5}$.

Solution.

1. Let $y = f^{-1}(x)$, i.e., $x = f(y)$. Then

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{3y^2 + 4}.$$

Alternatively, differentiate both sides of the equation $x = y^3 + 4y$ with respect to x, regarding x now as an implicit function of y. We get:

$$\frac{dx}{dy} = 3y^2 + 4 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3y^2 + 4}.$$

2. When $x = 5$, $y = f^{-1}(5) = 1$. (Check that $f(1) = 5$!) So,

$$\left.\frac{d}{dx} f^{-1}(x)\right|_{x=5} = \left.\frac{1}{3y^2 + 4}\right|_{y=1} = \frac{1}{7}.$$

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5.2 Higher Order Derivatives

Suppose that an object is moving along a coordinate line, and let t denote the time. parametrized by t. Let

$$s = s(t)$$

denote the coordinate of the object at time t. The *velocity* (or “instantaneous velocity”) of the object at time t is:

$$v(t) = s'(t).$$

The *acceleration* of the object at time t is:

$$a(t) = v'(t) = s''(t).$$

Notation Let $y = f(x)$.

1st derivative of f: $\frac{dy}{dx} = \frac{df}{dx} = f'(x)$

2nd derivative of f: $\frac{d^2y}{dx^2} = \frac{d^2f}{dx^2} = f''(x)$

$\vdots$ $\vdots$

n-th derivative of f: $\frac{d^ny}{dx^n} = \frac{d^nf}{dx^n} = f^{(n)}(x)$

Example 5.2.1.

1. $$\frac{d^n}{dx^n}(e^x) = e^x, \quad \frac{d^n}{dx^n}(a^x) = a^x \cdot (\ln a)^n.$$

2. $y = x^n, n \in \mathbb{N}$.

$$y^{(m)} = \begin{cases} n(n-1)(n-2)\cdots(n-m+1)x^{n-m}, & \text{if } m < n, \\ n(n-1)(n-2)\cdots 2 \cdot 1 = n!, & \text{if } m = n, \\ 0, & \text{if } m > n. \end{cases}$$

Example 5.2.2. Let y be defined implicitly by the equation $x^2 + y^2 + e^{xy} = 2$. Find y' and y'' at $x = 1$.

Solution. Differentiate both sides of the preceding equation with respect to x to get

$$2x + 2yy' + e^{xy}(y + xy') = 0. \quad ----(1)$$

Then differentiate both sides of the equation with respect to x one more time to get

$$2 + 2y'y' + 2yy'' + e^{xy}(y + xy')^2 + e^{xy}(2y' + xy'') = 0. \quad ----(2)$$

Inserting $x = 1, y = 0$ into Equations (1), (2), we have:

$$y'|_{x=1} = -2,$$
$$y''|_{x=1} = -10.$$

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Example 5.2.3. Suppose that $y = e^{\lambda x}$ satisfies $y'' - 2y' - 3y = 0$ (a "differential equation"). Find the constant λ.

Solution. $y = e^{\lambda x}$ implies that $y' = \lambda e^{\lambda x}$, which in turn implies $y'' = \lambda^2 e^{\lambda x}$.

Combining the preceding identities with the equation $y'' - 2y' - 3y = 0$, we have:

$$(\lambda^2 - 2\lambda - 3)e^{\lambda x} = 0.$$

Since $e^{\lambda x} \neq 0$ for all x,

$$\lambda^2 - 2\lambda - 3 = 0, \rightarrow \lambda = -1, 3.$$

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More generally, if $y = e^{\lambda x}$ solves

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

then

$$(a_n \lambda^{(n)} + a_{n-1} \lambda^{(n-1)} + \cdots + a_1 \lambda + a_0) e^{\lambda x} = 0,$$

$\Rightarrow$

$$a_n \lambda^{(n)} + a_{n-1} \lambda^{(n-1)} + \cdots + a_1 \lambda + a_0 = 0.$$

Exercise 5.2.1. Find constants λ such that $y = e^{\lambda} x$ satisfies $y''' - 2y'' - 3y' = 0$.
Answer: $\lambda = -1, 0, 3$.