MATH1520C University Mathematics for Applications **Fall 2021**

Chapter 1: Notation and Functions

Learning Objectives:
(1) Identify the domain of a function, and evaluate a function from an equation.
(2) Gain familiarity with piecewise functions.
(3) Study the vertical line test.
(4) Know how to form and use composite functions.

1.1 Set

- **Set** is a collection of objects (called **elements**)
  1. Order of elements does not matter. E.g. $\{1, 2, 3\} = \{3, 2, 1\}$.
  2. Representation of a set is not unique. E.g. $\{-2, 2\} = \{x \mid x^2 = 4\}$.
- $\in$: belongs to. If a is an element of A, we say that a belongs to A; denoted as $a \in A$.
- $\subset$: subset of. Let A, B be two sets such that $\forall a \in A$, $a \in B$. Then we say that A is a subset of B; denoted as $A \subset B$.

Remark. $A \subset B$ is sometimes written as $A \subseteq B$ to emphasize the fact that $A = B$ is a possibility. If $A \subset B$ but $A \neq B$, then A is said to be a *proper subset* (or a strict subset) of B, written as $A \Subset B$.
$A \subset B \Leftrightarrow B \supset A$: B is a *supset* of A.

Example 1.1.1.

1. $A = \{1, 2, 3\}$, $B = \{2, 3, 5, 7\}$, $C = \{1, 2, 3, 4, 5\}$. Then $A \subseteq C$ (in fact $A \Subset C$), $1 \in A$, but $1 \notin B$ and $B \not\subseteq C$.
2. $C =$ the set of all students studying at CUHK. $M =$ the set of all math major students currently studying at CUHK. Then $M \subseteq C$. You $\in C$.

Example 1.1.2. Some important number sets:

1. $\mathbb{N}$: the set of all natural numbers (positive integers) $=\{1, 2, 3, \ldots\}$.
2. $\mathbb{Z}$: the set of all integers $= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
3. $\mathbb{Q}$: the set of all rational numbers $=\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$.

4. $\mathbb{R}$: the set of all real numbers.

Remark. If the elements in a set can be ordered and the ordering are taken into account in the definition, then it is called an *ordered set*. E.g. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ may be viewed as ordered sets.

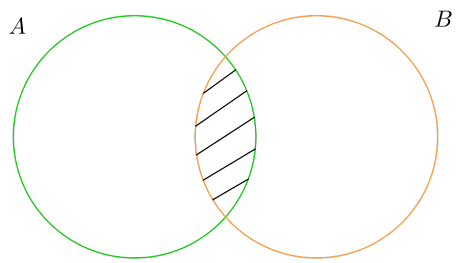
1.2 Intervals

- $[a, b]= \{x \mid a \leq x \leq b\}$. (closed interval)
- $(a, b)=\{x \mid a < x < b\}$. (open interval)
- $(a, b]= \{x \mid a < x \leq b\}$.
- $[a, \infty)$: the set of all real numbers x such that $a \leq x$.

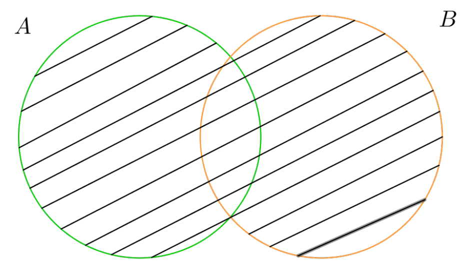
Drawing open/closed intervals on the real line:

1.3 Set operations

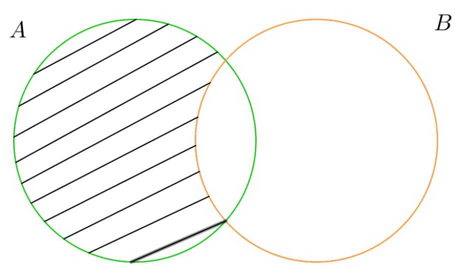
Let A, B be two sets:



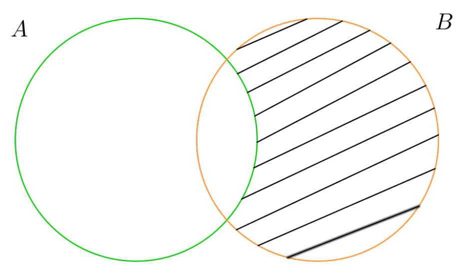
Intersection
$A \cap B = \{x \,|\, x \in A \;\text{ and }\; x \in B\}$



Union
$A \cup B = \{x \,|\, x \in A \;\text{ or }\; x \in B\}$



Relative complement of B in A
$A \backslash B = \{x \,|\, x \in A \;\text{ and }\; x \notin B\}$



Relative complement of A in B
$B \backslash A = \{x \,|\, x \in B \;\text{ and }\; x \notin A\}$

Remark. Alternate notation for $A \backslash B$: $A - B$.

Example 1.3.1.

1. Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $C = \{5\}$.
   $A \cap B = \{2, 3\}$, $A \cup B = \{1, 2, 3, 4\}$, $A \backslash B = \{1\}$, $B \backslash A = \{4\}$, $A \backslash C = A$.
2. $\mathbb{R} \backslash \{a\}$: the set of all real numbers x, except $x = a$.
3. $A \backslash B = A \backslash (A \cap B)$.

Exercise 1.3.1.

1. What are the meanings of the following sets
   (a) $(-\infty, a)$.
   (b) $\mathbb{R} \backslash \{1, 2, 3\}$
   (c) $\mathbb{R} \backslash [2, 3)$.
2. Show that $\mathbb{R} \backslash [1, \infty) = (-\infty, 1)$.

1.4 Functions

Definition 1.4.1. A **function** is a rule that assigns to EACH element x in a set A EXACTLY ONE element y in a set B. If the function is denoted by f, then we may write

$$f : A \to B.$$

The set A is called the domain of the function. The set B is called the codomain of f. The assigned elements in B is called the range of f.

$x \in A$ is the independent variable of f; $y = f(x) \in B$ is the dependent variable of f.

Given $a \in A$, $f(a) \in B$ is said to be the *value* of the function f at a. Given $S \subset A$,

$$f(S) := \{f(a) \,|\, a \in S\}$$

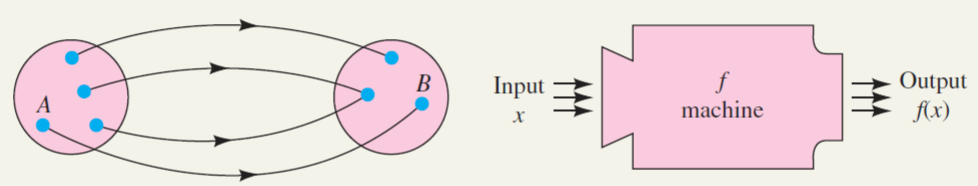
is said to be the *image* of S under f. In particular, the "range" of f, as defined above, is $f(A) \subset B$.

When the domain and range of a function are both sets of real numbers, the function is said to be a real-valued function of one variable, and we write

$$f : \mathbb{R} \to \mathbb{R}.$$

Most functions encountered in this course are real-valued functions of one variable. Unless otherwise specified, a function is a real-valued function of one variable in this course.

Remark. There is some ambiguity in the definition of "range" in math literature. See the Wiki article. A function $f : A \to B$ is also called a *map from A to B*; A is the *source* of f and B is the *target* of f.



(a) A function as a mapping (b) A function as a machine

Example 1.4.1. $f : [-1, 3) \to \mathbb{R}$ is defined by $f(x) = x^2 + 4$ (sometimes written as $y = x^2 + 4$). Then

$$f(0) = (0)^2 + 4 = 4.$$

domain = $[-1, 3)$, codomain = $\mathbb{R}$, range of f = $[4, 13)$.

Remark. If a function is given by a formula without domain specified, then assume domain = set of all x for which $f(x)$ is well defined, this domain is also called the natural domain of f.

Example 1.4.2. Find the natural domain of the functions.

1. $f(x) = \dfrac{1}{x-3}$.
2. $g(t) = \dfrac{\sqrt{3-2t}}{t^2+4}$.

Solution.

1. $\dfrac{1}{x-3}$ is not defined when its denominator $x - 3 = 0$, i.e. $x = 3$. So the domain is $\mathbb{R}\backslash\{3\}$.
2. The domain of $\sqrt{3-2t}$ consists of all x such that $3 - 2t \geq 0$, which implies that $t \leq \frac{3}{2}$. Hence the domain is $(-\infty, \frac{3}{2}]$.

■

Example 1.4.3. Let $f(x) = \dfrac{x^2-1}{x-1}$ and $g(x) = x + 1$. Can we say f and g are the same function?

Solution. No! The domain of $f(x)$ is $\mathbb{R}\backslash\{1\}$, the domain of $g(x)$ is $\mathbb{R}$.
Only when $x \neq 1$, $f(x) = g(x)$. ■

1.4.1 Vertical Line Test for Graph

A way to visualize a function is its graph. If f is a real-valued function of one variable, its graph consists of the points in the Cartesian plane $\mathbb{R}^2$ whose coordinates are the input-output pairs for f. In set notation, the graph is

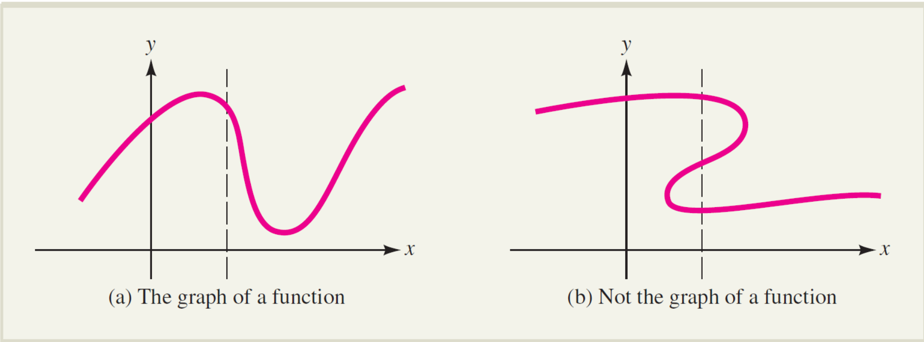
$$\Gamma(f) := \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R}, y = f(x)\}.$$

Review: Graphing a real-valued function of one variable: [HBSP] 1.2.

Example 1.4.4. linear functions; piecewise linear functions; quadractic functions, exponential and log functions, trig functions.

It is important to realize that not every curve is the graph of a function. For instance, suppose the circle $x^2 + y^2 = 5$ were the graph of some function $y = f(x)$. Then, since the points $(1, 2)$ and $(1, -2)$ both lie on the circle, we would have $f(1) = 2$ and $f(1) = -2$, contrary to the requirement that a function assigns one and only one value to each number in its domain. Geometrically, this happens because the vertical line $x = 1$ intersects the graph of the circle more than once. The vertical line test is a geometric rule for determining whether a curve is the graph of a function.

The Vertical Line Test A curve is the graph of a function if and only if no vertical line intersects the curve more than once:



(a) The graph of a function

(b) Not the graph of a function

1.4.2 Some Special Functions

Definition 1.4.2. A piecewise function is defined by more than one formula, with each individual formula defined on a subset of the domain.

Example 1.4.5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & \text{if } x < 0 \\ 2x, & \text{if } x \geq 0. \end{cases}$$

Then $f(-1) = 1$, $f(0) = 0$ and $f(1) = 2$.

Remark. If all the formulae involved in defining a piecewise function are linear, then the function is said to be *piecewise linear*. E.g. The function in the preceding example.

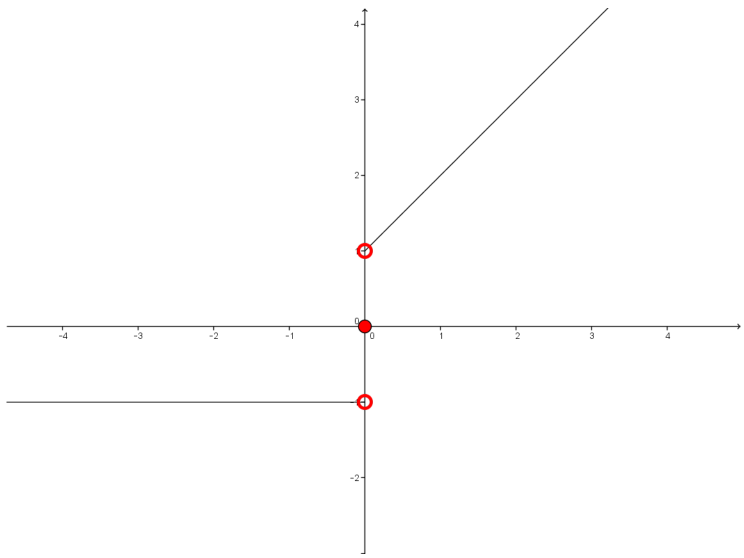
Example 1.4.6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -1, & \text{if } |x| \geq \pi \\ \sin x, & \text{if } |x| < \pi. \end{cases}$$

Example 1.4.7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

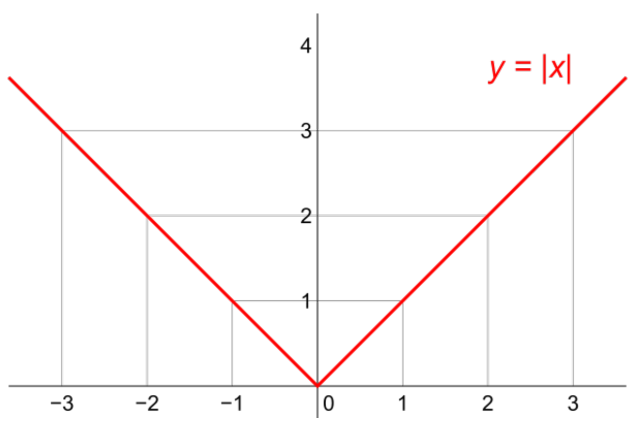
$$f(x) = \begin{cases} x+1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Then f is a piecewise (linear) function.



Example 1.4.8. The absolute value function

$$|x| := \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$



Example 1.4.9. Write $f(x) = 2x + |2 - x|$ as a piecewise function.

Solution. Note that $|2-x| = 2-x$ when $2-x \geq 0$, that is $x \leq 2$; and $|2-x| = x-2$ when $2-x < 0$, that is, $x > 2$. Hence $f(x) = 2x+2-x = x+2$ if $x \leq 2$, and $f(x) = 2x+x-2 = 3x-2$ if $x > 2$, or we can write

$$f(x) = \begin{cases} x+2 & \text{if } x \leq 2 \\ 3x-2 & \text{if } x > 2 \end{cases}.$$

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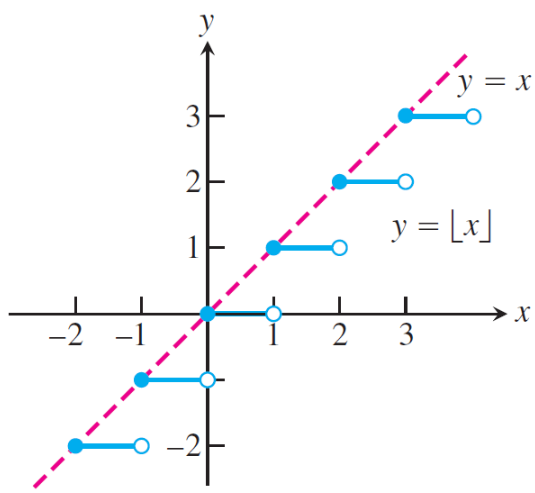
Example 1.4.10. Define the *floor function* as $\lfloor x \rfloor$ = the largest integer $\leq x$. Then $f(x) = \lfloor x \rfloor$ is a piecewise function.



Exercise 1.4.1. Define the *ceiling function* as $\lceil x \rceil$ = the smallest integer $\geq x$. Sketch the graph of $\lceil x \rceil$.

Exercise 1.4.2. Sketch the graph of

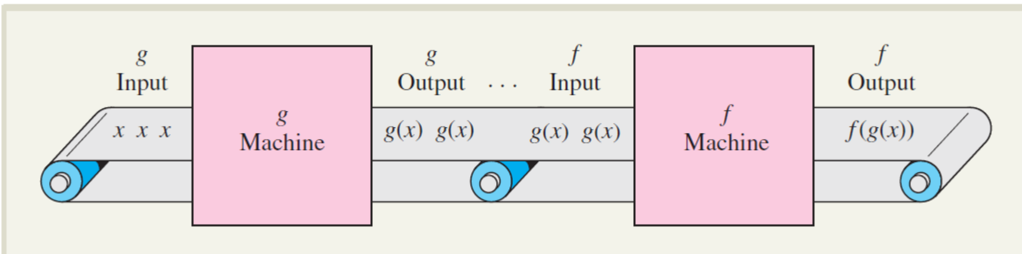
$$f(x) = \begin{cases} x-2, & \text{if } x > 1, \\ -1, & \text{if } 0 \leq x \leq 1, \\ x^2, & \text{if } x < 0. \end{cases}$$

1.5 Composition of functions

Definition 1.5.1. Given functions $f(u)$ and $g(x)$, the composition of f and g, denoted by $(f \circ g)(x)$, is a function of x formed by substituting $u = g(x)$ for u in the formula of $f(u)$, i.e.

$$(f \circ g)(x) = f(g(x)).$$

In the following figure, the definition of composite function is illustrated as an assembly line in which raw input x is first converted into a transitional product $g(x)$ that acts as input in f machine uses to produce $f(g(x))$.



Example 1.5.1. $f(x) = x^2 + 3x + 1$ and $g(x) = x + 1$.
Then

$$\begin{aligned}(f \circ g)(x) = f(g(x)) &= (g(x))^2 + 3(g(x)) + 1 = (x+1)^2 + 3(x+1) + 1 \\ &= (x^2 + 2x + 1) + (3x + 3) + 1 = x^2 + 5x + 5\end{aligned}$$

Similarly,

$$(g \circ f)(x) = g(f(x)) = f(x) + 1 = x^2 + 3x + 2.$$

Remark. In general, $(f \circ g)(x) \neq (g \circ f)(x)$.

Example 1.5.2. Suppose $f(x) = x^3 - 1$ and $h(x) = x - 1$, find $g(u)$ such that $f(x) = g(h(x))$.

Solution.

$$f(x) = x^3 - 1 = (x - 1 + 1)^3 - 1 = (x-1)^3 + 3(x-1)^2 + 3(x-1) = g(u),$$

where we define

$$g(u) = u^3 + 3u^2 + 3u.$$

■

Alternative solution (change of variables): Set $u = h(x)$ to be the new variable. Then $u = x - 1$ and x may be expressed in terms of the new variable u as $x = u + 1$. Plugging this into the formula for f, we have:

$$g(u) = f(x) = (u + 1)^3 - 1.$$

Example 1.5.3. Suppose $f(x) = (x - 5)^2 + \frac{3}{(x-5)^3}$, find $g(u)$ and $h(x)$ such that $f(x) = g(h(x))$.

Solution. The form of the given function is

$$f(x) = \square^2 + \frac{3}{\square^3},$$

where each box contains the expression $x - 5$. Thus $f(x) = g(h(x))$, where

$$g(u) = u^2 + \frac{3}{u^3} \text{ and } h(x) = x - 5.$$

■

Remark. There are many possible answers to the preceding problem, as $h(x)$ can be chosen quite arbitrarily. E.g. one may choose the new variable $u = h(x) = x - 1$, then $x = u + 1$ and

$$g(u) = f(x) = (u - 4)^2 + \frac{3}{(u-4)^3}.$$

Definition 1.5.2. A difference quotient for a function $f(x)$ is a composition function of the form

$$\frac{f(x + h) - f(x)}{h}$$

where h is a constant.

Difference quotients are used to compute the slope of a tangent line to the graph and define the derivative, a concept of central importance in calculus.

Example 1.5.4. Find the difference quotient of $f(x) = x^2 - 3x$.

Solution.

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{[(x+h)^2 - 3(x+h)] - [x^2 - 3x]}{h} \\ &= \frac{[x^2 + 2xh + h^2 - 3x - 3h] - [x^2 - 3x]}{h} \\ &= \frac{2xh + h^2 - 3h}{h} = 2x + h - 3.\end{aligned}$$

■

Geometric interpretation: As slopes of secant lines to the graph of f.

$h \to 0 \rightsquigarrow$ tangent lines. Slopes of tangent lines to the graph of $f \rightsquigarrow$ derivatives of f.

1.6 Modeling in Business and Economics

Example 1.6.1. A manufacturer can produce dinning room tables at a cost of \$200 each. The table has been selling for \$300 each, and at that price consumers have been buying 400 tables per month. The manufacturer is planning to raise the price of the table and estimates that for each \$1 increase in the price, 2 fewer tables will be sold each month. What price corresponds to the maximum profit, and what is the maximum profit?

Solution. Let x be the price.

$$\begin{aligned}
\text{Profit for one table} &= x - 200 \\
\text{Number of tables sold} &= 400 - 2(x - 300) = 1000 - 2x \\
\text{Total profit:} \quad f(x) &= (x - 200)(1000 - 2x) \\
&= -2x^2 + 1400x - 200000 \\
&= -2(x - 350)^2 + 45000
\end{aligned}$$

$f(x)$ is maximized when the manufacturer charges \$350 for each table. ■

Question: How to find max/min for general functions? Calculus helps!