MATH1050 Proof-writing Exercise 9 Index of Comments

1. **CR.**
   You are cramming two or more ideas into the same line, thus making the whole line unclear.
   Examples.
   (a) It is unclear what is meant by
   '*Let* $u = x + y \in \mathbb{Z}$.'
   '*Let* $u = x + y$' is one thing: you are defining u to be the sum $x + y$.
   '$x + y \in \mathbb{Z}$' is another thing: you are arguing that $x + y$ is an integer because of so-and-so.
   (b) It is unclear what is meant by
   '*Suppose it were true that* $p + q \geq (s + t)^2 \geq 0$.'
   '*Suppose it were true that* $p+q \geq (s+t)^2$' is one thing: you are stating some assumption which you probably intend to use in the subsequent argument.
   '$(s + t)^2 \geq 0$' is another thing: you are arguing that $(s + t)^2$ is a non-negative real number because of so-and-so.
   (c) It is unclear what is meant by
   '*There exists some* $k \in \mathbb{Z}$ *such that* $x + z = 4 = 3k$.'
   '$x + z = 4$' is one thing: you are asserting that the sum of x, z is 4.
   '*There exists some* $k \in \mathbb{Z}$ *such that* $x+z = 3k$' is another thing: you are elaborating the statement '$x+z$ *is divisible*' according to the definition of divisibility.
   (d) It is unclear why under the assumption '$g \circ f : A \longrightarrow C$ is surjective why the statement is true:
   '*For any* $z \in C$, *there exists some* $x \in A$ *such that* $z = g(f(x))$.'
   The problem is that this statement contains nothing about the function $g \circ f$.
   Surjectivity of $g \circ f$ is one thing. It gives:
   '*For any* $z \in C$, *there exists some* $x \in A$ *such that* $z = (g \circ f)(x)$.'
   Composition of functions is another thing. It tells us that $(g \circ f)(x) = g(f(x))$ (for the same $x \in A$).

   You should slow down, and write out the ideas one-by-one, expounding at most one idea within one sentence.

2. **DBC.**
   When you are going to dis-prove the statement
   $$\underbrace{(\forall x)(\forall y)...(\forall z)}_{\text{all universal quantifiers}} \quad [H(x, y, \cdots, z) \longrightarrow K(x, y, \cdots, z)]$$
   by giving a counter-example, you are actually proving the statement
   $$\underbrace{(\exists x)(\exists y)...(\exists z)}_{\text{all existential quantifiers}} \quad [H(x, y, \cdots, z) \wedge (\sim K(x, y, \cdots, z))].$$
   It is therefore wrong if you start your ' dis-proof by counter-example' argument with these words:
   - '*Let* x, y, z *be elements of blah-blah-blah. Suppose* $H(x, y, \cdots, z)$ *holds.*'

   You should just introduce, right-away, (what you believe to be) an appropriate example $x, y, \cdots, z$ (for which you believe $H(x, y, \cdots, z)$ holds and $K(x, y, \cdots, z)$ fails to hold.) Then verify, for the $x, y, \cdots, z$ that you have named, that $H(x, y, \cdots, z)$ indeed holds and $K(x, y, \cdots, z)$ indeed fails to hold.

3. **DEF.**
   Look up the definition. (For instance, the definition of surjectivity, or the definition of injectivity, or the definition of image set, or the definition of pre-image set.) You are not adhering to definition in your argument, or you have missed out key logical features in the definition. For this reason, your argument is deemed incomplete, if not altogether wrong.

4. **FCN.**
   (a) When introducing a function in an argument, the domain of the function, the range of the function, and the 'formula of definition' (or alternatively the graph of the function) must be specified clearly. Moreover, it must be made clear that a function is being introduced.
   It is not acceptable to just write something like:
   - '*Let* f *be the function given by* $f(x) = x^2$ *for any* $x \in \mathbb{R}$.'

- '*Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function given by $f(x) = x^2$, $x \in \mathbb{R}$.*'
- '*Lef $f(x) = x^2$.*'
- '*Let $A = \mathbb{R}, B = \mathbb{R}$, and $f(x) = x^2$.*'
- '*Let $A = \{0, 1\}, B = \{0, 1\}$. Take $f(0) = 0, f(1) = 1$.*'
- '*Let $f(0) = 1, f(1) = 2$.*'

Remember: it is not the reader's responsibility to fill in the blanks for you.

(b) When introducing a function by specifying its domain and range and its 'formula of definition' (which amounts to a verbal description of the graph of the function), care must be taken to ensure the function is 'well-defined', in the sense that Condition (E) and Condition (U) are both satisfied.

5. **INV.**

   No '$f^{-1}(x)$' unless f is known to be bijective.

6. **MA.**

   At least part of the assumptions is missing. But you are going to use these assumptions in the argument. The reader is not responsible to write out the missing assumptions for you, and will simply regard your argument as wrong when you are applying the 'missing assumptions'.

   While it is fine (and indeed nice) to state what you intend to argue for in an upcoming paragraph, when it indeed comes to that paragraph, you should make it clear what the assumptions are within the argument given in the paragraph. You must not leave it to the reader to guess what the assumption are.

7. **MS.**

   There is a missing step which you should not have skipped.

8. **NEG.**

   Your negation for the statement concerned is wrong. Very often, it is because of a mis-interpretation of the statement to be negated.

   Example. Consider the statement (M):

   > (M): '*Let $x, y, z \in \mathbb{N}$. Suppose $x + y$ is divisible by 3 and $y + z$ is divisible by 3. Then $x + z$ is divisible by 5.*'

   Expressed in the most formal way, with all the quantifiers explicitly presented, (M) reads:

   > (M): '*For any $x, y, z \in \mathbb{N}$, if $x + y$ is divisible by 3 and $y + z$ is divisible by 3 then $x + z$ is divisible by 5.*'

   So the negation $(\sim M)$ of (M) is:

   > $(\sim M)$: '*There exist some $x, y, z \in \mathbb{N}$ such that $x + y$ is divisible by 3 and $y + z$ is divisible by 3 and $x + z$ is not divisible by 5.*'

   To dis-prove (M) is to prove $(\sim M)$.

   Carefully read the notes on *Universal quantifier and existential quantifier*, *Statements with several quantifiers*, *Disproofs*.

9. **PC.**

   Punctuation and capitals should be used appropriately so as to indicate to the reader how the passage is to be read. Omissions may result in the reader being confused with the logic and/or the mathematical content in what you are writing.

10. **QF** (incorporating old **IA** and old **IE**).

    At least one of the things described below has happened:

    (a) You have misunderstood the logical structure and the mathematical content in

    i. '*there exists some $k \in \mathbb{Z}$ such that $a = kb$*',
    ii. '*$u = sv$ for some Gaussian integer s*'.

    Be aware of the information '*which depends on which*' carried in such a statement. (In the first example, k depends on a, b, and not the other way round. In the second example, s depends on u, v, and not other way round.)

    (b) You have missed the key idea of 'existence' in

    i. '*there exists some $k \in \mathbb{Z}$ such that $a = kb$*',
    ii. '*$u = sv$ for some Gaussian integer s*'.

    This matters in the logical structure of the argument. Your subsequent argument fails entirely because of it. (In the first example, k owes its existence from a, b in the context of the argument. In the second example, s owes its existence from u, v in the context of the argument.)

(c) This statement in which the existential quantifier '*there exist*' (or '*for some*') has appeared is not stated in an appropriate way: its logical structure has not been given due respect.

It is tempting to use '*where*' as a substitute for '*for some*'. The problem is that this leads to various for misinterpretation (including the misreading of such a '*where*' as '*for any*'.)

(d) old **IA.**

This is redundant and confusing use of the quantifier 'for any'.

You have already introduced, say, a concrete object a, earlier. After that point, it is wrong to write '*for any* a *...*', because it would confuse the reader on whether you are talking about the same a, or you want to introduce another object which you quite careless and wrongly label as a. In fact, if you are doing things correctly, you can be (as you should be) referring to the same object a.

(e) old **IE.**

This is redundant and confusing use of the quantifiers 'there exist', 'for some'.

You have already introduced, say, a concrete object c, earlier. After that point, it is wrong to write '*for some* c *...*', '*there exists* c *...*', because it would confuse the reader on whether you are talking about the same c, or you want to introduce another object which you quite careless and wrongly label as c.

When you want to deduce a statement of the form

'*there exists some* c *such that* c *satisfies blah-blah-blah*',

simply name that c (which you know will satisfy blah-blah-blah) and proceed to verify blah-blah-blah. Having finished this process, it is needless and pointless to write

'*hence there exists some* c *such that* c *satisfies blah-blah-blah*',

(This looks strange, but 'existence of something' is demonstrated by presenting that 'something' in concrete terms, not by repeating the word 'exists'.)

(f) Never use 'for any', 'there exist' in a way like what you see below:

- '*for any* $f(x) \in B$ *...* ',
- '*there exists some* $f(x) \in B$ *...* ',
- '*for any* $y = f(x) \in B$ *...*',
- '*there exists some* $y = f(x) \in B$ *...*'.

The problem is: Are you introducing x, or f, or $f(x)$, or y, or what? What is the variable in the predicate which is supposed appear after such words 'for any ...', 'there exists ...'? x? Or f? $f(x)$? y? Or what?