MATH1050 Proof-writing Exercise 9 (Answers and selected solution)

1. **Solution.**

   Denote the interval $(0, +\infty)$ by I. Let $f : I \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \frac{1}{2}\left(x - \frac{1}{x}\right)$ for any $x \in I$.

   (a) Pick any $x, w \in I$. Suppose $f(x) = f(w)$.
   Then $\frac{1}{2}\left(x - \frac{1}{x}\right) = \frac{1}{2}\left(w - \frac{1}{w}\right)$.
   We have $x^2w - w = w^2x - x$.
   Therefore $xw(x - w) = x^2w - w^2x = w - x$. Hence $(xw + 1)(x - w) = 0$.
   Since $x, w \in I$, we have $x > 0, w > 0$ and $xw + 1 > 0$. Therefore $x - w = 0$. Hence $x = w$.
   It follows that f is injective.

   (b) Pick any $y \in \mathbb{R}$. Take $x = y + \sqrt{y^2 + 1}$. Note that $x \in I$.
   We have $f(x) = \frac{1}{2}\left(x - \frac{1}{x}\right) = \frac{1}{2}\left(y + \sqrt{y^2+1} - \frac{1}{y + \sqrt{y^2+1}}\right)$
   $= \frac{1}{2}\left[y + \sqrt{y^2+1} - \frac{y - \sqrt{y^2+1}}{(y + \sqrt{y^2+1})(y - \sqrt{y^2+1})}\right] = \frac{1}{2}(y + \sqrt{y^2+1} + y - \sqrt{y^2+1}) = y.$
   It follows that f is surjective.

2. ——

3. **Answer.**

   (a) Yes.

   (b) Yes.

4. **Answer.**

   (a) No.

   (b) No.

5. (a) **Solution.**

   i. Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions.
   Suppose $g \circ f$ is surjective.
   Pick any $z \in C$.
   Since $g \circ f$ is surjective, there exists some $x \in A$ such that $z = (g \circ f)(x)$.
   Define $y = f(x)$. We have $y \in B$.
   For the same $x \in A$, $y \in B$ and $z \in C$, we have $g(y) = g(f(x)) = (g \circ f)(x) = z$.
   It follows that g is surjective.

   ii. Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions.
   Suppose $g \circ f$ is injective.
   Pick any $x, w \in A$. Suppose $f(x) = f(w)$.
   Then $g(f(x)) = g(f(w))$.
   Note that $(g \circ f)(x) = g(f(x))$ and $(g \circ f)(w) = g(f(w))$.
   Then $(g \circ f)(x) = (g \circ f)(w)$.
   Since $g \circ f$ is injective, we have $x = w$.
   It follows that f is injective.

   (b) ——

6. ——

7. ——