# MATH1050 Proof-writing Exercise 9

**Advice.**

- Study the Handouts *Surjectivity and injectivity*, *Surjectivity and injectivity for 'nice' real-valued functions of one real variable*, *Surjectivity and injectivity for 'simple' complex-valued functions of one complex variable*, *Compositions, surjectivity and injectivity* before answering the questions.
- Besides the handouts mentioned above, Questions (2), (3) of Exercise 9 are also relevant.

1. Denote the interval $(0, +\infty)$ by $I$. Let $f : I \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{1}{2}\left(x - \dfrac{1}{x}\right)$ for any $x \in I$.

   (a) Verify that $f$ is injective, with reference to the definition of injectivity.

   (b) Verify that $f$ is surjective, with reference to the definition of surjectivity.

   **Remark.** Do not use any results from the calculus of one real variable. They are not needed in the first place.

2. Let $f : (0, +\infty) \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4} \cos\left(\dfrac{1}{x\sqrt{x}}\right)$ for any $x \in (0, +\infty)$.

   (a) Verify that $f$ is not injective, with reference to the definition of injectivity.

   (b) i. Verify that $\left|\dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}\right| \leq 1$ for any $x \in (0, +\infty)$.

   **Remark.** A very simple answer can be obtained without using calculus.

   ii. Apply the previous part, or otherwise, to verify that $f$ is not surjective, with reference to the definition of surjectivity.

3. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = z^2\bar{z}$ for any $z \in \mathbb{C}$.

   (a) Is $f$ injective? Prove your answer, with reference to the definition of injectivity.

   (b) Is $f$ surjective? Prove your answer, with reference to the definition of surjectivity.

4. Let $f : \mathbb{C}\backslash\{0\} \longrightarrow \mathbb{C}\backslash\{0\}$ be the function defined by $f(z) = \dfrac{z}{\bar{z}}$ for any $z \in \mathbb{C}\backslash\{0\}$.

   (a) Is $f$ injective? Prove your answer, with reference to the definition of injectivity.

   (b) Is $f$ surjective? Prove your answer, with reference to the definition of surjectivity.

5. (a) Prove each of the statements below:

   i. *Let $A, B, C$ be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose $g \circ f$ is surjective. Then $g$ is surjective.*

   ii. *Let $A, B, C$ be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose $g \circ f$ is injective. Then $f$ is injective.*

   (b) Let $I, J, K$ be sets, and $\alpha : I \longrightarrow J$, $\beta : J \longrightarrow K$, $\gamma : K \longrightarrow I$ be functions. Suppose $\gamma \circ \beta \circ \alpha$, $\alpha \circ \gamma \circ \beta$ are both injective. Further suppose $\beta \circ \alpha \circ \gamma$ is surjective.
   Prove that each of the functions $\alpha, \beta, \gamma$ is both surjective and injective.

6. Consider each of the statements below. Dis-prove it by giving an appropriate argument.

   (a) *Let $A, B, C$ be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose $g \circ f$ is surjective. Then $f$ is surjective.*

   (b) *Let $A, B, C$ be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose $g \circ f$ is injective. Then $g$ is injective.*

7. Let $A$ be an $(m \times n)$-matrix with real entries. Define the function $L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by $L_A(\mathbf{x}) = A\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$.
   (The function $L_A$ is called the **linear transformation defined by matrix multiplication from the left by** $A$.)

   (a) Verify that $L_A(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha L_A(\mathbf{x}) + \beta L_A(\mathbf{y})$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, for any $\alpha, \beta \in \mathbb{R}$.

   (b) Verify that $L_A$ is injective iff $\mathcal{N}(A) = \{\mathbf{0}\}$.

   **Remark.** Recall that $\mathcal{N}(A)$ is the null space of $A$, given by $\mathcal{N}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$.

   (c) Verify that $L_A$ is surjective iff $\mathcal{C}(A) = \mathbb{R}^m$.

   **Remark.** Recall that $\mathcal{C}(A)$ is the column space of $A$, given by $\mathcal{C}(A) = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n\}$.