MATH1050 Proof-writing Exercise 7 (Answers and selected solution)

1. (a) **Answer.**
   There exist some $x, y, z \in \mathbb{N}$ such that $x + y$, $y + z$ are divisible by 3 and$x + z$ is not divisible by 3.

   (b) **Solution.**
   Take $x = z = 1$, $y = 2$.
   We have $x, y, z \in \mathbb{N}$.
   Note that $x + y = y + z = 3 = 1 \cdot 3$. We have $1 \in \mathbb{Z}$.
   Then, by definition, $x + y$, $y + z$ are divisible by 3.
   Note that $x + z = 2$. We verify that 2 is not divisible by 3:

   Suppose 2 were divisible by 3.
   Then there would exist some $k \in \mathbb{Z}$ such that $2 = 3k$.
   For the same k, we would have $k = \dfrac{2}{3}$. Then k is not an integer.
   Contradiction arises.

2. (a) ——

   (b) ——

   (c) **Solution.**
   Denote by M the statement below:

   M: Let n be a positive integer, and ζ be a complex number. Suppose ζ is an n^2-th root of unity. Then ζ^2 is an n-th root of unity.

   The negation of M reads:

   $\sim M$: There exist some positive integer n and some complex number ζ such that ζ is an n^2-th root of unity and ζ^2 is not an n-th root of unity.

   We verify $\sim M$:

   - Take $n = 3$, $\zeta = \cos\left(\dfrac{2\pi}{9}\right) + i \sin\left(\dfrac{2\pi}{9}\right)$.
     $\zeta^{3^2} = \zeta^9 = \cos\left(9 \cdot \dfrac{2\pi}{9}\right) + i \sin\left(9 \cdot \dfrac{2\pi}{9}\right) = \cos(2\pi) + i \sin(2\pi) = 1.$
     Then ζ is a n^2-th root of unity.
     $\zeta^2 = \cos\left(2 \cdot \dfrac{2\pi}{9}\right) + i \sin\left(2 \cdot \dfrac{2\pi}{9}\right) = \cos\left(\dfrac{4\pi}{9}\right) + i \sin\left(\dfrac{4\pi}{9}\right).$
     $(\zeta^2)^3 = \cos\left(3 \cdot \dfrac{4\pi}{9}\right) + i \sin\left(3 \cdot \dfrac{4\pi}{9}\right) = \cos\left(\dfrac{4\pi}{3}\right) + i \sin\left(\dfrac{4\pi}{3}\right) = -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2} i \neq 1.$
     Then ζ^2 is not an n-th root of unity.

3. (a) **Solution.**
   Denote by M the statement below:

   M: Suppose A, B, C be sets. Then $A \backslash (C \backslash B) \subset A \cap B$.

   The negation of M reads:

   $\sim M$: There exist some sets A, B, C such that $A \backslash (C \backslash B) \not\subset A \cap B$.

   We verify $\sim M$:

   - Regard $0, 1, 2$ as distinct objects.
     Let $A = \{0, 1\}$, $B = \{1\}$, $C = \{2\}$.
     We have $A \cap B = B = \{1\}$, $C \backslash B = C = \{2\}$, $A \backslash (C \backslash B) = A = \{0, 1\}$.
     Note that $0 \in A \backslash (C \backslash B)$ and $0 \notin A \cap B$.
     Hence $A \backslash (C \backslash B) \not\subset A \cap B$.

   (b) ——

4. ——

5. ——