MATH1050 Proof-writing Exercise 4

Advice.

- Study the Handout *Examples of proofs-by-contradiction* before answering the questions.
- Besides the handout mentioned above, Questions (10a), (11a) of Exercise 4 are also suggestive on what it takes to give a correct argument with the proof-by-contradiction method. First and foremost is to remember that the assumptions needed within the argument must be stated clearly.
- When giving an argument, remember to adhere to definition, always.

1. Apply proof-by-contradiction to justify the statements below:
   (a) *Let a, b be complex numbers. Suppose $a^4 + a^3b + a^2b^2 + ab^3 + b^4 \neq 0$. Then at least one of a, b is non-zero.*
   (b) *Let a, b be real numbers. Suppose $ab > 1$. Then $a^2 + 4b^2 > 4$.*
   (c) *Let ζ be a complex number. Suppose that $|\zeta| \leq \varepsilon$ for any positive real number ε. Then $\zeta = 0$.*

2. (a) Explain the phrase *prime number* by giving its appropriate definition.
   (b) Apply proof-by-contradiction to justify the statement below:
   - *Let p, q be prime numbers. Suppose p, q are positive and $p \neq q$. Then p is not divisible by q.*

3. *You may tacitly assume the result $(*)$, stated below:*
   $(*)$ *Suppose u, v are rational numbers. Then $u + v, u - v, uv$ are rational numbers. Moreover, if $v \neq 0$ then u/v is a rational number.*

   Apply proof-by-contradiction to justify the statement below, with the help of $(*)$ where appropriate:
   - *Let x be a positive real number, r be a positive rational number, and n be an integer greater than 1. Suppose x is an irrational number. Then $\sqrt[n]{x + r}$ is an irrational number.*

4. Apply proof-by-contradiction to justify the statement below:
   - *Let a, b be real numbers. Suppose $|a| \leq 1$ and $|b| \leq 1$. Then $\sqrt{1 - a^2} + \sqrt{1 - b^2} \leq 2\sqrt{1 - (a + b)^2/4}$.*

5. We introduce/recall the definitions on *absolute extremum* for real-valued functions of one real variable:

   Let I be an interval, and $h : I \longrightarrow \mathbb{R}$ be a real-valued function of one real variable.
   - *h is said to* **attain absolute maximum on** *I if there exists some $p \in I$ such that for any $x \in I$, the inequality $h(x) \leq h(p)$ holds.*
     The number $h(p)$ is called the **absolute maximum value of** *h* **on** *I.*
   - *h is said to* **attain absolute minimum on** *I if there exists some $p \in I$ such that for any $x \in I$, the inequality $h(x) \geq h(p)$ holds.*
     The number $h(p)$ is called the **absolute minimum value of** *h* **on** *I.*

   Apply proof-by-contradiction to justify the statements below, with direct reference to the definitions on absolute extremum. Do not use any results from the calculus of one real variable.
   (a) *Suppose $h : [0, 1) \longrightarrow \mathbb{R}$ is the real-valued function of one real variable defined by $h(x) = x^2$ for any $x \in [0, 1)$. Then h does not attain absolute maximum on $[0, 1)$.*
   (b) *Suppose $h : [0, +\infty) \longrightarrow \mathbb{R}$ is the real-valued function of one real variable defined by $h(x) = \sqrt{x}$ for any $x \in [0, +\infty)$. Then h does not attain absolute maximum on $[0, +\infty)$.*
   (c) *Suppose $h : [0, +\infty) \longrightarrow \mathbb{R}$ is the real-valued function of one real variable defined by $h(x) = \dfrac{1}{1 + x^2}$ for any $x \in [0, +\infty)$. Then h does not attain absolute minimum on $[0, +\infty)$.*
   (d) *Let a, b be real numbers, with $a < b$, and $h : (a, b) \longrightarrow \mathbb{R}$ be a real-valued function of one real variable. Suppose h is strictly increasing. Then h does not attain absolute maximum on (a, b).*
   Remark. You need the definition for strict monotonicity.

6. For each $n \in \mathbb{N}\backslash\{0\}$, define $A_n = \displaystyle\sum_{j=1}^{n} \frac{1}{j}$, $B_n = \displaystyle\sum_{k=1}^{n} \frac{1}{2k}$, $C_n = \displaystyle\sum_{k=1}^{n} \frac{1}{2k - 1}$.
   (a) i. Prove that $B_n = \dfrac{1}{2}A_n$ and $C_n = A_{2n} - \dfrac{1}{2}A_n$ for any $n \in \mathbb{N}\backslash\{0\}$.
   ii. Prove that $C_n - B_n \geq \dfrac{1}{2}$ for any $n \in \mathbb{N}\backslash\{0, 1\}$.
   (b) By applying proof-by-contradiction, or otherwise, prove that $\{A_n\}_{n=1}^{\infty}$ does not converge in $\mathbb{R}$.
   Remark. Take for granted any 'standard' results on limits of sequences, covered in MATH1010.