MATH1050 Exercise 12

1. Define $J = \left\{ ((s,t),(u,v)) \;\middle|\; \begin{array}{l} s,t,u,v \in \mathbb{N}, \text{ and} \\ [s < u \text{ or } (s = u \text{ and } t \leq v)] \end{array} \right\}$, and $T = (\mathbb{N}^2, \mathbb{N}^2, J)$. Note that $J \subset (\mathbb{N}^2)^2$.

   (a) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement '*T is reflexive*'. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   > We verify the statement '______ (I) ______'.
   > ____ (II) ____ $p \in \mathbb{N}^2$ . ____ (III) ____ $s, t \in \mathbb{N}$ ______ (IV) ______ .
   > We have $s = s$ ____ (V) ____ .
   > Then by the definition of $J$, we have ____ (VI) ____ .
   > It follows that $T$ is reflexive.

   (b) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement '*T is anti-symmetric*'. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   > We verify the statement '______ (I) ______'.
   > Pick any $p, q \in \mathbb{N}^2$. ______ (II) ______ .
   > There exist some $s, t, u, v \in \mathbb{N}$ such that $p = (s, t)$ and $q = (u, v)$.
   > Since $(p, q) \in J$, we have $(s < u$ or $(s = u$ and $t \leq v))$.
   > Then $s \leq u$ ____ (III) ____ $(s < u$ ____ (IV) ____ $t \leq v)$, by the Law of Distribution for conjunction and disjunction.
   > In particular, $s \leq u$.
   > Since ____ (V) ____ , we also deduce ____ (VI) ____ by modifying the argument above.
   > Then $s \leq u$ and ____ (VII) ____ . Therefore $s = u$.
   > Now recall that $(s < u$ or $(s = u$ and $t \leq v))$.
   > Then $s = u$ and ____ (VIII) ____ . In particular, $t \leq v$.
   > Similarly, we deduce that ____ (IX) ____ .
   > Then $t \leq v$ ____ (X) ____ $v \leq t$. Therefore ____ (XI) ____ .
   > Then $s = u$ and $t = v$. Hence ____ (XII) ____
   > It follows that $T$ is anti-symmetric.

   (c) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement '*T is transitive*'. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

We verify the statement ‘ ______(I)______ ’.

Pick any $p, q, r \in \mathbb{N}^2$. Suppose $(p, q) \in J$ and $(q, r) \in J$.

* (Case 1.) ______(II)______ . Then $(p, r) = (q, r) \in J$.
* (Case 2.) Suppose $q = r$. Then ______(III)______ .
* (Case 3.) Suppose ______(IV)______ .

  ______(V)______ .

  Since ______(VI)______ , we have $s < u$ or ($s = u$ and $t \leq v$).

  Since $p \neq q$, ‘$s = u$ and $t = v$’ is ______(VII)______ .

  Then we have $s < u$ ______(VIII)______ ($s = u$ ______(IX)______ $t < v$).

  Since ______(X)______ , we can also deduce that ______(XI)______ , by modifying the argument above.

  Now we have [$s < u$ or ($s = u$ and $t < v$)] and [$u < w$ or ($u = w$ and $v < x$)].

  * (Case 3a.) Suppose $s < u$ and $u < w$. Then ______(XII)______ . Therefore $s < w$ or ($s = w$ and $t < x$).
  * (Case 3b.) ______(XIII)______ . Then $s < u = w$. Therefore $s < w$ or ($s = w$ and $t < x$).
  * (Case 3c.) ______(XIV)______ . Then $s = u < w$. Therefore $s < w$ or ($s = w$ and $t < x$).
  * (Case 3d.) ______(XV)______ . Then $s = u = w$ and $t < v < x$. Therefore $s < w$ or ($s = w$ and $t < x$).

  Then in any case $s < w$ or ($s = w$ and $t < x$). Hence ______(XVI)______ .

Hence in any case, $(p, r) \in J$.
It follows that $T$ is transitive.

(d) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement ‘*$T$ is strongly connected*’. (*The ‘underline’ for each blank bears no definite relation with the length of the answer for that blank.*)

We verify the statement ‘ ______(I)______ ’.

______(II)______ .

There exist some $s, t, u, v \in \mathbb{N}$ such that $p = (s, t)$ and $q = (u, v)$.
We have $s < u$ or $s = u$ or $s > u$.

* (Case 1.) Suppose $s < u$. Then ______(III)______ . Therefore $(p, q) \in J$.
* (Case 2.) ______(IV)______ . We have $t \leq v$ ______(V)______ $v \leq t$.
  * (Case 2a.) ______(VI)______ . Then $s = u$ and $t \leq v$. Therefore $s < u$ or ($s = u$ and $t \leq v$). Hence $(p, q) \in J$.
  * (Case 2b.) ______(VII)______ . Then $u = s$ and $v \leq t$. Therefore $u < s$ or ($u = s$ and $v \leq t$). Hence ______(VIII)______ .
* (Case 3.) ______(IX)______ . Then ______(X)______ . Therefore ______(XI)______ .

Hence in any case, ______(XII)______ .
It follows that $T$ is strongly connected.

(e) According to the previous parts, $T$ is a total ordering.

From now on, we write $p \leq_{\text{lex}} q$ exactly when $(p, q) \in J$.
Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement '*$T$ is well-order relation in $\mathbb{N}^2$*'. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

We verify the statement '___(I)___ subset $B$ of ___(II)___, if ___(III)___, then ___(IV)___'.

___(V)___. Take some $p \in B$. ___(VI)___ such that $p = (u, v)$.

- Define the set $B' = \{x \in \mathbb{N} : (x, y) \in B$ ___(VII)___ $y \in \mathbb{N}\}$.

  $B'$ is a ___(VIII)___ according to its definition.

  By definition of $B'$, ___(IX)___. Then $B'$ is non-empty.

  By the Well-ordering Principle for Integers , $B'$ has a ___(X)___. Denote it by $s$.

- Define $B'' = \{y \in \mathbb{N} : (s, y) \in B\}$.

  $B''$ is a subset of $\mathbb{N}$ according to its definition.

  Recall that $s \in B'$. Then by definition of $B'$, ___(XI)___ $w \in \mathbb{N}$ ___(XII)___.

  By definition of $B''$, $w \in B''$.

  Therefore ___(XIII)___.

  By ___(XIV)___, $B''$ has a least element.

  Denote it by $t$.

- We verify that $(s, t)$ is the least element of $B$ with respect to $T$.

  According to the definition for the notion of least element, we verify the statement '___(XV)___'

  Pick any $q \in B$. By definition, $q \in \mathbb{N}^2$.
  Then there exist some $x, y \in \mathbb{N}$ such that $q = (x, y)$.
  By definition of $B'$, ___(XVI)___, we have $x \in B'$.
  Then, since ___(XVII)___, we have $s \leq x$.
  Note that $s < x$ ___(XVIII)___ $s = x$.

  * (Case 1.) Suppose $s < x$. Then $s < x$ ___(XIX)___ ($s = x$ ___(XX)___ $t \leq y$).
    Therefore $(s, t) \leq_{\text{lex}} (x, y) = q$.
  * (Case 2.) ___(XXI)___. Then, by definition of $B''$, since $(s, y) = (x, y) \in B$, we have ___(XXII)___.
    Now, since $t$ is a least element of $B''$, we have ___(XXIII)___
    Then $s = x$ and $t \leq y$. Therefore $s < x$ or ($s = x$ and $t \leq y$).
    Then ___(XXIV)___.

  Hence, in any case, $(s, t) \leq_{\text{lex}} q$.

It follows that $T$ is a well-order relation in $\mathbb{N}^2$.

2. Define the relation $R = (\mathbb{R}, \mathbb{R}, P)$ in $\mathbb{R}$ by $P = \{(x, y) \in \mathbb{R}^2 :$ There exists some $n \in \mathbb{N}$ such that $y = 2^n x\}$.
   (a) Verify that $R$ is a partial ordering in $\mathbb{R}$.
   (b) Is $R$ a total ordering in $\mathbb{R}$? Why?
   (c) Is $R$ an equivalence relation in $\mathbb{R}$? Why?

3. Let $R = (\mathbb{R}, \mathbb{R}, G)$ be the relation in $\mathbb{R}$ defined by $G = \{(x, y) \mid x \in \mathbb{R}$ and $y \in \mathbb{R}$ and $|y - x| \leq 1\}$.
   (a) Verify that $R$ is reflexive.
   (b) Is $R$ symmetric? Justify your answer.

(c) Is $R$ anti-symmetric? Justify your answer.

(d) Is $R$ transitive? Justify your answer.

(e) Is $R$ an equivalence relation in $\mathbb{R}$? Is $R$ a partial ordering in $\mathbb{R}$? Or neither? Why?

4. Let $A = \{\varphi \mid \varphi : \mathbb{N} \longrightarrow \mathbb{R} \text{ is a function and } \varphi(0) = 0.\ \}$. Define the relation $R = (A, A, H)$ in $A$ by $H = \{(\varphi, \psi) \in A^2 : \varphi(n+1) - \varphi(n) \leq \psi(n+1) - \psi(n) \text{ for any } n \in \mathbb{N}\}$.

   (a) Verify that $R$ is reflexive.

   (b) Verify that $R$ is transitive.

   (c) Is $R$ a partial ordering in $A$? Justify your answer.

   (d) Is $R$ an equivalence relation in $A$? Justify your answer.

5. Define the relation $R = (\mathbb{Z}, \mathbb{Z}, G)$ in $\mathbb{Z}$ by $G = \{(x, y) \in \mathbb{Z}^2 : \text{There exist some } m, n \in \mathbb{N} \text{ such that } y = mx + n.\}$.

   (a) Verify that $R$ is reflexive.

   (b) Verify that $R$ is transitive.

   (c) Verify that $R$ is not a partial ordering in $\mathbb{Z}$.

   (d) Is $R$ is an equivalence relation in $\mathbb{Z}$? Justify your answer.

6. Let $J = [0, 1)$, $K = (0, 1]$.
Define $H = \{(x, y) \mid x \in J \text{ and } y \in K \text{ and } x + y = 1\}$, and $h = (J, K, H)$. Note that $H \subset J \times K$.
Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement '*$J$ is of cardinality equal to $K$*'. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

> We verify the statement $(E)$: 'for any $x \in J$, ____(I)____ '.
>
> Pick any ____(II)____ . Take ____(III)____ . By definition, we have $x + y = x + (1 - x) = 1$.
>
> Since ____(IV)____ , we have $0 \leq x < 1$.
>
> Since $x \geq 0$, we have ____(V)____ .
>
> Since $x < 1$, we have ____(VI)____ .
>
> Then $0 < y \leq 1$. Therefore ____(VII)____ .
>
> Hence ____(VIII)____ .
>
> We verify the statement $(U)$: 'for any $x \in J$, for any $y, z \in K$, ____(IX)____ '.
>
> ____(X)____ . Suppose ____(XI)____ .
>
> ____(XII)____ , we have $x + y = 1$ . Then $y = 1 - x$.
>
> Since $(x, z) \in H$, we have ____(XIII)____ . Then ____(XIV)____ .
>
> Therefore ____(XV)____ .
>
> We verify the statement $(S)$: ' ____(XVI)____ such that ____(XVII)____ '.
>
> ____(XVIII)____ . Take $x = 1 - y$. By definition, we have $x + y = (1 - y) + y = 1$.
>
> Since $y \in K$, we have ____(XIX)____ .
>
> Since ____(XX)____ , we have $x = 1 - y < 1 - 0 = 1$.
>
> Since $y \leq 1$, we have ____(XXI)____ .
>
> Then $0 \leq x < 1$. Therefore $x \in J$.
>
> Hence ____(XXII)____ .
>
> We verify the statement $(I)$: ' ____(XXIII)____ if $(x, y) \in H$ and $(w, y) \in H$ ____(XXIV)____ '.
>
> Pick any $x, w \in J$, $y \in K$. ____(XXV)____ .
>
> ____(XXVI)____ , we have $x + y = 1$. Then $x = 1 - y$.
>
> Since $(w, y) \in H$, we have ____(XXVII)____ . Then $w = 1 - y$.
>
> Therefore ____(XXVIII)____ .
>
> By $(E), (U)$, ____(XXIX)____ . By $(S)$, ____(XXX)____ . By $(I)$, ____(XXXI)____ .
> Hence $h$ is a bijective function from $J$ to $K$ with graph $H$.
> It follows that ____(XXXII)____ .

7. Let $J = [0, 1)$, $L = (0, 1)$, $M = [0, +\infty)$, $N = (0, +\infty)$.
   (a) By writing down appropriate bijective functions, verify that $J \sim M$ and $L \sim N$.
   (b) By writing down an appropriate bijective function, verify that $L \sim \mathbb{R}$.

8. Let $D = \{z \in \mathbb{C} : |z| < 1\}$, $H = \{w \in \mathbb{C} : \mathsf{Im}(w) > 0\}$.
Define $F = \left\{ (z, w) \,\middle|\, z \in D \text{ and } w \in H \text{ and } w = \dfrac{z + i}{iz + 1} \right\}$, and $f = (D, H, F)$. Note that $F \subset D \times H$.

(a) Is $f$ a function? Justify your answer.

(b) Is it true that $D \sim H$? Justify your answer.

9. Let $I = (0, +\infty)$, $J = [-1, 1]$.

   (a) Prove that $\dfrac{1}{a+1} \in J$ for any $a \in I$.

   (b) Define the function $g : I \longrightarrow J$ by $g(x) = \dfrac{1}{x+1}$ for any $x \in I$. Is $g$ injective? Justify your answer.

   (c) Apply the Schröder-Bernstein Theorem to prove that $I \sim J$.

10. Let $A = [-1, 1]$, $B = (-4, -2] \cup [2, 4)$.

    (a) Name one injective function from $A$ to $B$, if there is any at all, and verify that it is indeed an injective function from $A$ to $B$.

    (b) Apply the Schröder-Bernstein Theorem, or otherwise, to prove that $A$ is of cardinality equal to $B$.

11.$^\diamondsuit$ Let $A = [1010, 1050] \backslash \{1030\}$ and $B = (2040, 2050) \cup ([2060, +\infty) \cap \mathbb{Q})$.

    (a) Name one injective function from $A$ to $B$, if there is any at all, and verify that it is indeed an injective function from $A$ to $B$.

    (b) Apply the Schröder-Bernstein Theorem to prove that $A \sim B$.

12. Let $A, B, C$ be sets. Prove the statements below:

    (a) $A \sim A$.

    (b) *Suppose $A \sim B$. Then $B \sim A$.*

    (c) *Suppose $A \sim B$ and $B \sim C$. Then $A \sim C$.*

    (d) $A \lesssim A$.

    (e) *Suppose $A \lesssim B$ and $B \lesssim C$. Then $A \lesssim C$.*

13. (a) Let $A, B, C, D$ be sets, and $f : A \longrightarrow C$, $g : B \longrightarrow D$ be functions. Define the function $f \times g : A \times B \longrightarrow C \times D$ by $(f \times g)(x, y) = (f(x), g(y))$ for any $x \in A$, for any $y \in B$.

       i. Suppose $f, g$ are surjective. Verify that $f \times g$ is surjective.

       ii. Suppose $f, g$ are injective. Verify that $f \times g$ is injective.

       iii. Suppose $f, g$ are bijective. Verify that $f \times g$ is bijective.

    (b) Let $A, B, C, D$ be sets. Suppose $A \sim C$ and $B \sim D$. Prove that $A \times B \sim C \times D$.

       **Remark.** Hence the statement below holds:

       Let $A, B$ be sets. Suppose $A \sim B$. Then $A^2 \sim B^2$.

14. Let $A$ be a non-empty set. Define the function $\chi : \mathfrak{P}(A) \longrightarrow \mathsf{Map}(A, \{0, 1\})$ by $\chi(S) = \chi_S^A$ for any $S \in \mathfrak{P}(A)$.

    Here, for any subset $S$ of $A$, $\chi_S^A : A \longrightarrow \{0, 1\}$ is the characteristic function of $S$ in $A$, defined by

    $$\chi_S^A(x) = \begin{cases} 1 & \text{if} \quad x \in S, \\ 0 & \text{if} \quad x \in A \backslash S. \end{cases}$$

    (a)$^\clubsuit$ Verify that $\chi$ is surjective.

    (b)$^\clubsuit$ Verify that $\chi$ is injective.

    (c) Is it true that $\mathfrak{P}(A) \sim \mathsf{Map}(A, \{0, 1\})$? Justify your answer.

15. Define the functions $\sigma, \tau : \mathbb{N} \longrightarrow \mathbb{N}$ by $\sigma(n) = 2n$, $\tau(n) = 2n + 1$ for any $n \in \mathbb{N}$.

    Let $B$ be a non-empty set. Define the function $f : \mathsf{Map}(\mathbb{N}, B) \longrightarrow (\mathsf{Map}(\mathbb{N}, B))^2$ by $f(\varphi) = (\varphi \circ \sigma, \varphi \circ \tau)$ for any $\varphi \in \mathsf{Map}(\mathbb{N}, B)$.

    (a)$^\clubsuit$ Verify that $f$ is surjective.

    (b)$^\clubsuit$ Verify that $f$ is injective.

    (c) Is it true that $\mathsf{Map}(\mathbb{N}, B) \sim (\mathsf{Map}(\mathbb{N}, B))^2$? Justify your answer.

16.$^\clubsuit$ *In this question, we are going to give another proof for Cantor's Theorem on the power set of any given set.*

    (a) Let $A$ be a set, and $f : A \longrightarrow \mathfrak{P}(A)$ be a function. Define $C_f = \{x \in A : x \notin f(x)\}$. (Note that $C_f \in \mathfrak{P}(A)$.)

       i. Dis-prove the statement '*there exists some $z \in A$ such that $f(z) = C_f$.*

       ii. Hence deduce that $f$ is not surjective.

       **Remark.** The set $C_f$ is called **Cantor's diagonal set for the function $f$.**

    (b) Apply the above results to prove Cantor's Theorem on the power set of any given set.