MATH1050 Exercise 11

1. *You are not required to justify your answer in this question. You are only required to give one correct answer for each ordered pair, although there may be different correct answers.*

   Let $A = [0, +\infty)$ and G, H be the subsets of $\mathbb{R}^2$ defined respectively by $G = \{(x, x) \mid x > 0\}$, $H = \{(x, y) \mid x \geq 0 \text{ and } y > 0 \text{ and } x^2 + y^2 = 1\}$.

   Name some appropriate ordered pairs $(s, t), (u, v) \in A^2$, if such exist, for which the ordered triple $(A, A, (G \cup H \cup \{(s, t), (u, v)\}))$ is a reflexive and symmetric relation in A.

2. Define the relation $S = (\mathbb{R}, \mathbb{R}, E)$ by $E = \{(x, y) \in \mathbb{R}^2 : x - y = a \text{ for some } a \in \mathbb{Q}\}$.

   Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement '*S is an equivalence relation in* $\mathbb{R}$'. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

- [We want to verify the reflexivity of S.]

We verify the statement '________(I)________':

________(II)________ .

Note that ________(III)________ , and $0 \in \mathbb{Q}$. Then ______(IV)______ , by the definition of S.

It follows that S is reflexive.

- [We want to verify the symmetry of S.]

We verify the statement '________(V)________':

Pick any $x, y \in \mathbb{R}$. ________(VI)________ .

Then ________(VII)________ such that $x - y = a$.

Note that ________(VIII)________ .

Since ________(IX)________ , we have $-a \in \mathbb{Q}$.

Now we have $y - x = -a$ and ______(X)______ . Then ______(XI)______ , by the definition of S.

It follows that S is symmetric.

- [We want to verify the transitivity of S.]

We verify the statement '________(XII)________':

Pick any ________(XIII)________ . Suppose ________(XIV)________ .

Since $(x, y) \in E$, there exists some $a \in \mathbb{Q}$ such that ________(XV)________ .

________(XVI)________ .

Note that $x - z = (x - y) + (y - z) = a + b$.

________(XVII)________ .

Now we have $x - z = a + b$, and $a + b \in \mathbb{Q}$. Then ______(XVIII)______ , by the definition of S.

It follows that S is transitive.

Since S is reflexive, symmetric and transitive, S is an equivalence relation.

3. Define the relation $S = (\mathbb{R}, \mathbb{R}, G)$ by $G = \{(x, y) \in \mathbb{R}^2 : x - y > xy\}$.

   Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement '*S is not reflexive, and S is not symmetric, and S is not transitive*'. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

- [We verify that S is not reflexive.]

We verify the statement '______(I)______':

______(II)______. Note that $x_0 \in \mathbb{R}$.

Also note that ______(III)______ $= 0 \leq 0 =$ ______(IV)______. Therefore '$x_0 - x_0 > x_0 \cdot x_0$' is false.

Hence ______(V)______.

It follows that S is not reflexive.

- [We verify that S is not symmetric.]

We verify the statement '______(VI)______':

Take ______(VII)______. Note that $x_0, y_0 \in \mathbb{R}$.

Note that $x_0 - y_0 =$ ______(VIII)______ $= x_0 y_0$. Then ______(IX)______.

Also note that $y_0 - x_0 = -1 \leq 0 = y_0 x_0$. Then '______(X)______' is false.

Therefore $(y_0, x_0) \notin G$.

Hence for the same $x_0, y_0 \in \mathbb{R}$, ______(XI)______ simultaneously.

It follows that S is not symmetric.

- [We verify that S is not transitive.]

We verify the statement '______(XII)______':

Take $x_0 = -2$, $y_0 = 3$, ______(XIII)______. Note that $x_0, y_0, z_0 \in \mathbb{R}$.

Note that ______(XIV)______ Then $(x_0, y_0) \in G$.

Also note that $y_0 - z_0 = 5 > -6 = y_0 z_0$. Then ______(XV)______.

Finally, note that ______(XVI)______. Then '$x_0 - z_0 > x_0 z_0$' is false.

Therefore ______(XVII)______.

Hence for the same $x_0, y_0, z_0 \in \mathbb{R}$, ______(XVIII)______ simultaneously.

It follows that S is not transitive.

4. *You are not required to justify your answer in this question.*
   Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
   Which of the sets below are partitions of A? Which not?

$$\Omega = \{\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}\}, \qquad \Xi = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}\},$$
$$\Sigma = \{\{0, 1\}, \{2, 3, 4, 5, 5\}, \{7, 8, 9\}\}, \qquad \mathrm{T} = \{\{0, 1\}, \{2, 3, 4\}, \{4, 5, 6\}, \{6\}, \{7, 8, 9\}\},$$
$$\Pi = \{\{1, 3, 5, 7, 9\}, \{0, 2, 4, 6, 8\}\}, \qquad \Upsilon = \{\{0, 1\}, \{2, 3, 4\}, \{5, 6, 7, 8, 9\}, \{4, 3, 2\}\},$$
$$\Gamma = \{\emptyset, \{0, 2\}, \{1, 3, 5\}, \{4, 6, 8\}, \{7, 9\}\}, \qquad \Delta = \{\{0, 1\}, \{2, 3, 4\}, \{5, 6, 7, 8, 9, 10\}\}.$$

5. *You are not required to justify your answer in this question.*
   Let $A = \{0, 1, 2, 3, 4, 5\}$, and Ω be the partition of A given by $\Omega = \{\{0\}, \{1, 3, 5\}, \{2, 4\}\}$.
   Write down all the elements of the graph E_Ω of the equivalence relation R_Ω in A induced by the partition Ω.

6. Define the relation $S = (\mathbb{C}, \mathbb{C}, F)$ in $\mathbb{C}$ by $F = \{(\zeta, \xi) \in \mathbb{C}^2 : \zeta^2 - \xi^2 = ai \text{ for some } a \in \mathbb{R}\}$.
   (a) Verify that S is reflexive.
   (b) Verify that S is symmetric.
   (c) Verify that S is an equivalence relation in $\mathbb{C}$.

7. Write $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$. Define the relation $R = (\mathbb{C}^*, \mathbb{C}^*, E)$ in $\mathbb{C}^*$ by $E = \left\{(\zeta, \eta) \in (\mathbb{C}^*)^2 : \dfrac{\eta}{\zeta} \in \mathbb{R}^*\right\}$.

   (a) Verify that R is reflexive.

   (b) Verify that R is transitive.

   (c) Verify that R is an equivalence relation in $\mathbb{C}^*$.