MATH1050 Exercise 10 (Answers and selected solution)

1. **Solution.**

   (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^4 - 4x^2$ for any $x \in \mathbb{R}$.

   i. We verify that f is not injective:
      - Take $x_0 = 0$, $w_0 = 2$. We have $x_0, w_0 \in \mathbb{R}$ and $x_0 \neq w_0$. Also, $f(x_0) = 0 = f(w_0)$.

   ii. We verify that f is not surjective:
      - Take $y_0 = -5$.
        Pick any $x \in \mathbb{R}$. We have $f(x) = x^4 - 4x^2 = (x^2 - 2)^2 - 4 \geq -4 > -5$. Then $f(x) \neq -5$.
        Hence, for any $x \in \mathbb{R}$, $f(x) \neq y_0$.

   (b) Let $x \in (\sqrt{2}, +\infty)$. $x^4 - 4x^2 = (x^2 - 2)^2 - 4 > 0 - 4 = -4$.

   (c) Let $g : (\sqrt{2}, +\infty) \longrightarrow (-4, +\infty)$ be the function defined by $g(x) = x^4 - 4x^2$ for any $x \in (\sqrt{2}, +\infty)$.

   i. Pick any $x, w \in (\sqrt{2}, +\infty)$. Suppose $g(x) = g(w)$. Then $x^4 - 4x^2 = w^4 - 4w^2$.
   Therefore $(x - w)(x + w)(x^2 + w^2) = (x^2 - w^2)(x^2 + w^2) = 4(x^2 - w^2) = 4(x - w)(x + w)$.
   Then $(x - w)(x + w)(x^2 + w^2 - 4) = 0$.
   Note that $x \geq \sqrt{2} > 0$ and $w \geq \sqrt{2} > 0$. Then $x + w > 0$ and $x^2 + w^2 - 4 > 0$.
   Then $x = w$.
   It follows that g is injective.

   ii. Pick any $y \in (-4, +\infty)$. Note that $y + 4 > 0$. Then $\sqrt{y+4}$ is well-defined and $2 + \sqrt{4+y} > 2$. Therefore $\sqrt{2 + \sqrt{4+y}}$ is well-defined and $\sqrt{2 + \sqrt{4+y}} > \sqrt{2}$.
   Take $x = \sqrt{2 + \sqrt{4+y}}$. Note that $x \in (\sqrt{2}, +\infty)$.
   We have $g(x) = x^4 - 4x^2 = (\sqrt{2 + \sqrt{4+y}})^4 - 4(\sqrt{2 + \sqrt{4+y}})^2 == (2 + \sqrt{4+y})^2 - 4(2 + \sqrt{4+y}) + 4 - 4 = [(2 + \sqrt{4+y}) - 2]^2 - 4 = (4 + y) - 4 = y$.
   It follows that g is surjective.

   iii. Since g is both injective and surjective, g is bijective. Its inverse function $g^{-1} : (-4, +\infty) \longrightarrow (\sqrt{2}, +\infty)$ is given by $g^{-1}(y) = \sqrt{2 + \sqrt{4+y}}$ for any $y \in (-4, +\infty)$.

   Remark. Although f and g have the same 'formula of definition', one is bijective and the other is not. So when talking about a function, be aware of its domain and its range, and don't just look at its 'formula of definition'.

2. **Answer.**

   (a) $J = (1, +\infty)$.

   (b) $f^{-1}(y) = \dfrac{1}{4}\left(\ln\left(\dfrac{y+1}{y-1}\right)\right)^2$ for any $y \in J$.

3. **Answer.**

   (a) ——

   (b) The inverse function $f^{-1} : \mathbb{C} \longrightarrow \mathbb{C}$ of the function f is given by $f^{-1}(z) = \bar{z}$ for any $z \in \mathbb{C}$.
   Comment. The conjugate of the conjugate of a complex number is the complex number itself.

4. **Answer.**

   (a) ——

   (b) i. ——

   ii. ——

   iii. The 'formula of definition' for inverse function $f^{-1} : \mathbb{C}\backslash\{a/c\} \longrightarrow \mathbb{C}\backslash\{-d/c\}$ of the function f is given by $f^{-1}(\zeta) = \dfrac{d\zeta - b}{-c\zeta + a}$ for any $\zeta \in \mathbb{C}\backslash\{a/c\}$.

5. **Answer.**

   (a) (I) Suppose $y \in f(S)$
   (II) there exists some $x \in S$ such that $y = f(x)$
   (III) $y = f(x) = 2x^4 - 4 \geq 2 \cdot 1^4 - 4 = -2$

(IV) Since $x \leq 2$

(V) Take $x = \sqrt[4]{\dfrac{y+4}{2}}$

(VI) $x = \sqrt[4]{\dfrac{y+4}{2}} \geq 1$

(VII) Since $y \leq 28$, we have $\dfrac{y+4}{2} \leq 16$

(VIII) $x = \sqrt[4]{\dfrac{y+4}{2}} \leq 2$

(IX) $f(x) = 2x^4 - 4 = 2\left(\sqrt[4]{\dfrac{y+4}{2}}\right)^4 - 4 = 2 \cdot \dfrac{y+4}{2} - 4 = y + 4 - 4 = y$

(X) $y \in f(S)$

(b) (I) Suppose $x \in f^{-1}(U)$

(II) there exists some $y \in U$ such that $y = f(x)$

(III) $y \in U$

(IV) $2x^4 - 4 = f(x) = y \leq 4$

(V) Suppose $x \in [-\sqrt{2}, \sqrt{2}]$

(VI) Define $y = f(x)$

(VII) $y = f(x) = 2x^4 - 4 \leq 4$

(VIII) $y = f(x) = 2x^4 - 4 \geq -6$

(IX) and

(X) $x \in f^{-1}(U)$

6. **Answer.**

(a) $(p, q) = (0, 0)$ or $(p, q) = (0, 1)$. $(s, t) = (1, \tau)$, provided that $-2 \leq \tau < 2$.

(b) $(p, q) = (1, 1)$ and $(s, t) = (2, 1)$. *Alternative answer*: $(p, q) = (1, 2)$ and $(s, t) = (2, 2)$.

(c) $(m, n) = (0, 0)$. $(p, q) = (1, 1)$ or $(p, q) = (1, 2)$.