MATH1050 Exercise 10

1. (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^4 - 4x^2$ for any $x \in \mathbb{R}$.
   i. Is f injective? Justify your answer.
   ii. Is f surjective? Justify your answer.

   (b) Verify that for any $x \in (\sqrt{2}, +\infty)$, $x^4 - 4x^2 > -4$.

   (c) Let $g : (\sqrt{2}, +\infty) \longrightarrow (-4, +\infty)$ be the function defined by $g(x) = x^4 - 4x^2$ for any $x \in (\sqrt{2}, +\infty)$.
   i. Is g injective? Justify your answer.
   ii. Is g surjective? Justify your answer.
   iii. Is g bijective? If *yes*, also write down the 'formula of definition' for its inverse function.

2. *You are not required to prove your answers in this question.*

   The function $f : (0, +\infty) \longrightarrow J$, given by $f(x) = \dfrac{e^{\sqrt{x}} + e^{-\sqrt{x}}}{e^{\sqrt{x}} - e^{-\sqrt{x}}}$ for any $x \in (0, +\infty)$ is known to be a bijective function from $(0, +\infty)$ to the set J.

   (a) Express the set J explicitly as an interval.

   (b) Write down the explicit 'formula of definition' for the inverse function f^{-1} of the function f.

3. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = \bar{z}$ for any $z \in \mathbb{C}$.

   (a) Verify that f is bijective.

   (b) Write down the 'formula of definition' of the inverse function of f.

4. Let $a, b, c, d \in \mathbb{C}$. Suppose $c \neq 0$ and $ad - bc \neq 0$.

   (a) Prove that for any $z \in \mathbb{C}$, $\dfrac{az + b}{cz + d} \neq \dfrac{a}{c}$.

   (b) Define the function $f : \mathbb{C} \backslash \{-d/c\} \longrightarrow \mathbb{C} \backslash \{a/c\}$ by $f(z) = \dfrac{az + b}{cz + d}$ for any $z \in \mathbb{C} \backslash \{-d/c\}$.
   i. Verify that f is injective.
   ii. Verify that f is surjective.
   iii. Write down the 'formula of definition' of the inverse function of f.

5. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = 2x^4 - 4$ for any $x \in \mathbb{R}$.

   (a) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the equality $f([1, 2]) = [-2, 28]$. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

Write $S = [1, 2]$.

- [We want to verify the statement (†): '*for any* y, *if* $y \in f(S)$ *then* $y \in [-2, 28]$.']

  Pick any y. ______(I)______ .

  Then by the definition of $f(S)$, ______(II)______ .

  For the same x, since $x \in S$, we have $1 \leq x \leq 2$.

  Since $x \geq 1$, we have ______(III)______ .

  ______(IV)______ we have $y = f(x) = 2x^4 - 4 \leq 2 \cdot 2^4 - 4 = 28$.

  Therefore $-2 \leq y \leq 28$.

  Hence $y \in [-2, 28]$.

- [We want to verify the statement (‡): '*for any* y, *if* $y \in [-2, 28]$ *then* $y \in f(S)$.']

  Pick any y. Suppose $y \in [-2, 28]$. Then $-2 \leq y \leq 28$.

  [We want to verify that for this y, there exists some $x \in S$ such that $y = f(x)$.]

  ______(V)______ .

  We verify that $x \in S$:

  * Since $y \geq -2$, we have $\dfrac{y+4}{2} \geq 1$. Then ______(VI)______ .

    ______(VII)______ . Then ______(VIII)______ .

    Therefore $1 \leq x \leq 2$. Hence $x \in [1, 2] = S$.

  For the same x, we have ______(IX)______ .

  Then, for the same x, y, we have $x \in S$ and $y = f(x)$.

  Hence by the definition of $f(S)$, ______(X)______ .

It follows that $f(S) = [-2, 28]$.

(b) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the equality $f^{-1}([-6, 4]) = [-\sqrt{2}, \sqrt{2}]$. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

Write $U = [-6, 4]$.

- [We want to verify the statement (†): '*for any* x*, if* $x \in f^{-1}(U)$ *then* $x \in [-\sqrt{2}, \sqrt{2}]$*.*']

  Pick any x. $\underline{\qquad\text{(I)}\qquad}$.

  Then by the definition of $f^{-1}(U)$, $\underline{\qquad\qquad\text{(II)}\qquad\qquad}$

  For the same y, since $\underline{\quad\text{(III)}\quad}$, we have $-6 \leq y \leq 4$.

  Since $y \geq -6$, we have $2x^4 - 4 = f(x) = y \geq -6$. Then $x^4 \geq -1$. (This provides no information other than re-iterating '$x \in \mathbb{R}$'.)

  Since $y \leq 4$, we have $\underline{\qquad\qquad\text{(IV)}\qquad\qquad}$. Then $x^4 \leq 4$. Since $x \in \mathbb{R}$, we have $-\sqrt{2} \leq x \leq \sqrt{2}$.

  Then $x \in [-\sqrt{2}, \sqrt{2}]$.

- [We want to verify the statement (‡): '*for any* x*, if* $x \in [-\sqrt{2}, \sqrt{2}]$ *then* $x \in f^{-1}(U)$*.*']

  Pick any x. $\underline{\qquad\text{(V)}\qquad}$. Then $-\sqrt{2} \leq x \leq \sqrt{2}$.

  [We want to verify that for this x, there exists some $y \in U$ such that $y = f(x)$.]

  $\underline{\qquad\text{(VI)}\qquad}$. We verify that $y \in U$:

  * Since $-\sqrt{2} \leq x \leq \sqrt{2}$, we have $x^4 \leq 4$. Then $\underline{\qquad\text{(VII)}\qquad}$.

    Since $x \in \mathbb{R}$, we have $x^4 \geq 0 \geq -1$. Then $\underline{\qquad\text{(VIII)}\qquad}$.

    Therefore $-6 \leq y \leq 4$. Hence $y \in [-6, 4] = U$.

  Then, for the same x, y, we have $y = f(x)$ $\underline{\quad\text{(IX)}\quad}$ $y \in U$.

  Hence by the definition of $f^{-1}(U)$, $\underline{\qquad\text{(X)}\qquad}$.

It follows that $f^{-1}(U) = [-\sqrt{2}, \sqrt{2}]$.

6. *You are not required to justify your answers in this question. In each part, you are only required to give one correct answer, although there are different correct answers.*

   (a) Let $A = (-1, 1]$, $B = [-2, 2)$, $G = \{(x, x) \mid x \leq 0\}$, $H = \{(x, x + 1) \mid x \geq 0\}$ and $F = (A \times B) \cap (G \cup H)$.

   Name some appropriate $(p, q), (s, t) \in A \times B$, if such exist, for which the ordered triple $(A, B, (F \backslash \{(p, q)\}) \cup \{(s, t)\})$ is a function from A to B.

   (b) Let $A = [0, 2]$, $G = \{(x, x^2) \mid 0 \leq x \leq 1\}$, $H = \{(x, 3 - x) \mid 1 \leq x < 2\}$ and $F = A^2 \cap (G \cup H)$.

   Name some appropriate $(p, q), (s, t) \in A^2$, if such exist, for which the ordered triple $(A, A, (F \backslash \{(p, q)\}) \cup \{(s, t)\})$ is an injective function from A to A.

   (c) Let $A = [0, +\infty)$ and E, F be the subsets of $\mathbb{R}^2$ defined respectively by $E = \{(x, x^{-1}) \mid 0 < x \leq 1\}$, $F = \{(x, 2x^{-2}) \mid x \geq 1\}$.

   Name some appropriate $(m, n), (p, q) \in A^2$, if such exist, for which the ordered triple $(A, A, (E \cup F \cup \{(m, n)\}) \backslash \{(p, q)\})$ is a surjective function from A to A.