MATH1050 Exercise 9 (Answers and selected solution)

1. **Solution.**

   (a) M is formally formulated as: 'For any set A, for any functions $f, g : A \longrightarrow A$, the equality $g \circ f = f \circ g$ as functions holds.'
   Hence $\sim M$ reads: 'There exist some set A, some functions $f, g : A \longrightarrow A$ such that $g \circ f \neq f \circ g$ as functions.'

   (b) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions defined by $f(x) = \dfrac{x^2}{1+x^2}$, $g(x) = x - 1$ for any $x \in \mathbb{R}$.

   i. For any $x \in \mathbb{R}$, we have

$$\begin{aligned}(g \circ f)(x) = g(f(x)) &= g\left(\frac{x^2}{1+x^2}\right) = \frac{x^2}{1+x^2} - 1 = -\frac{1}{1+x^2},\\ (f \circ g)(x) = f(g(x)) &= f(x-1) = \frac{(x-1)^2}{1+(x-1)^2}.\end{aligned}$$

   ii. We have $(g \circ f)(0) = -1$ and $(f \circ g)(0) = \dfrac{1}{2}$. Hence $(g \circ f)(0) \neq (f \circ g)(0)$.

   iii. There exists some $x_0 \in \mathbb{R}$, namely, $x_0 = 0$, such that $(g \circ f)(x_0) \neq (f \circ g)(x_0)$. Hence it is not true that $g \circ f = f \circ g$ as functions.

   (c) Let $A = \{0, 1\}$.
   Define $f, g : A \longrightarrow A$ by $f(0) = f(1) = 0$, $g(0) = g(1) = 1$.
   $g \circ f : A \longrightarrow A$ is given by $(g \circ f)(0) = g(f(0)) = g(0) = 1$, $(g \circ f)(1) = g(f(1)) = g(0) = 1$.
   $g \circ f : A \longrightarrow A$ is given by $(f \circ g)(0) = f(g(0)) = f(1) = 0$, $(f \circ g)(1) = f(g(1)) = f(1) = 0$.
   Take $x_0 = 0$. We have $(g \circ f)(x_0) = 1$ and $(f \circ g)(x_0) = 0$. Then $(g \circ f)(x_0) \neq (f \circ g)(x_0)$.
   Therefore $g \circ f \neq f \circ g$ as functions.

   Remark. Let A be a set. (This set is fixed in our subsequent discussion.) Suppose $f, g : A \longrightarrow A$ are two functions from the set A to A itself.

   - When we want to verify that $g \circ f$, $f \circ g$ are the same function from A to A, we have to verify that for any $x \in A$, $(g \circ f)(x) = (f \circ g)(x)$.
   - To verify that $g \circ f$, $f \circ g$ are not the same function from A to A, we check that there exists some $x_0 \in A$ such that $(g \circ f)(x_0) \neq (f \circ g)(x_0)$. Hence we have to name an appropriate x_0 and show that $(g \circ f)(x_0) \neq (f \circ g)(x_0)$.

2. **Answer.**

   (a) (I) Pick any
   Alternative answer. Let
   Alternative answer. Suppose
   Alternative answer. Assume
   Alternative answer. Take any

   (II) $x = (y+1)^{\frac{3}{5}}$

   (III) $f(x) = x^{\frac{5}{3}} - 1 = \left[(y+1)^{\frac{3}{5}}\right]^{\frac{5}{3}} - 1 = (y+1) - 1 = y$

   (IV) f is surjective.

   (b) (I) $x, w \in \mathbb{R}$

   (II) Suppose
   Alternative answer. Assume

   (III) $f(x) + 1 = f(w) + 1$

   (IV) $(w^{\frac{5}{3}})^{\frac{3}{5}} = w$

   (V) f is injective

3. **Answer.**

(a) (I) Take
Alternative answer. Let
Alternative answer. Define
Alternative answer. Pick
Alternative answer. Suppose
Alternative answer. Assume
(II) for any $x \in \mathbb{R}$
(III) Suppose
Alternative answer. Assume
(IV) there existed some $x_0 \in \mathbb{R}$ such that $f(x_0) = y_0$.
(V) 0
(VI) $\left(x_0 - \frac{1}{2}\right)^2 + \frac{3}{4} \geq 0 + \frac{3}{4}$
(VII) f is not surjective.

(b) (I) $w_0 = 2$.
(II) $x_0 \neq w_0$.
(III) $f(w_0) = \frac{2}{2^2+1} = \frac{2}{5}$
(IV) $f(w_0)$
(V) f is not injective

4. ——

5. **Answer.**

   (a) No. Note that $f(0) = f(1)$.

   (b) No. Note that $f(x) \neq 1$ for any $x \in \mathbb{R}$.

6. **Answer.**

   (a) ——

   (b) No.

7. **Answer.**

   (a) No.

   (b) No.

8. (a) *Hint.* The crucial step is to apply Triangle Inequality to obtain

$$f(b) = b^3\left(1 + \frac{p}{b} + \frac{q}{b^2} + \frac{r}{b^3}\right) \geq b^3\left(1 - \frac{|p|}{b} - \frac{|q|}{b^2} - \frac{|r|}{b^3}\right).$$

   (b) *Hint.* For each $\gamma \in \mathbb{R}$, apply the result in the previous to the function $f_\gamma : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f_\gamma(x) = \frac{g(x) - \gamma}{A}$ for any $x \in \mathbb{R}$.

9. **Answer.**

   (a) $1, 2$.

   (b) $0, 1$.

   (c) It is the empty set.

   (d) $1, 2$.

   (e) $2, 3, 4$.

   (f) 2.

   (g) $0, 1, 2, 3, 4$.

   (h) $1, 2$.

   (i) $2, 3, 4$.

10. **Answer.**

   (a) $\alpha = -1$.

   (b) $\beta = 1$, $\gamma = \frac{5}{4}$.

   (c) $\delta = -1$, $\varepsilon = \frac{5}{3}$.

   (d) $\zeta = -\sqrt{2} + 1$, $\eta = \sqrt{2} + 1$.

(e) $\theta = -\sqrt{2}+1,\ \kappa = -\frac{\sqrt{5}}{2}+1,\ \lambda = \frac{\sqrt{5}}{2}+1,\ \mu = \sqrt{2}+1.$

(f) $\nu = -\frac{1}{\sqrt{2}}+1,\ \xi = \frac{1}{\sqrt{2}}+1.$

(g) $\rho = -\frac{1}{\sqrt{2}}+1,\ \sigma = \frac{1}{\sqrt{2}}+1.$

(h) $\tau = -\sqrt{2}+1,\ \varphi = -\frac{1}{\sqrt{2}}+1,\ \psi = \frac{1}{\sqrt{2}}+1,\ \omega = \sqrt{2}+1.$

11. **Answer.**

(a) $\alpha = 0, \beta = 3, \gamma = 1, \delta = -1.$

(b) $\varepsilon = -2,\ \zeta = -0.25,\ \eta = -1,\ \theta = 0,\ \kappa = 1.$

12. **Answer.**

(a) $a = 2, b = 3.$

(b) $1, 4.$

(c) $\alpha = 0,\ \beta = 1,\ \gamma = 2,\ \delta = 4.$