MATH1050 Exercise 9

1. (a) Consider the statement M below:
   - *Let A be a set, and $f, g : A \longrightarrow A$ be functions. The equality $g \circ f = f \circ g$ as functions holds.*

   Write down the negation $\sim M$ of the statement M.

   (b) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions defined by $f(x) = \dfrac{x^2}{1+x^2}$, $g(x) = x - 1$ for any $x \in \mathbb{R}$.

   i. Compute the respective 'formulae of definition' of the functions $g \circ f$, $f \circ g$ explicitly.
   ii. Choose some $x_0 \in \mathbb{R}$ so that $(g \circ f)(x_0) \neq (f \circ g)(x_0)$.
   iii. Is it true that $g \circ f = f \circ g$ as functions? Justify your answer.

   Remark. Hence we have dis-proved the statement M with a counter-example. (Why?)

   (c) Define $A = \{0, 1\}$. Prove that there exist some functions $f, g : A \longrightarrow A$ such that $g \circ f \neq f \circ g$ as functions.

   Remark. Hence we have dis-proved the statement M with another counter-example. (Why?)

2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^{\frac{5}{3}} - 1$ for any $x \in \mathbb{R}$.

   (a) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the surjectivity of f. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   > [We want to verify the statement '*for any $y \in \mathbb{R}$, there exists some $x \in \mathbb{R}$ such that $y = f(x)$.*']
   >
   > _____(I)_____ $y \in \mathbb{R}$.
   > Take _____(II)_____ .
   > Note that $x \in \mathbb{R}$.
   > Also note that _____(III)_____ .
   > It follows that _____(IV)_____ .

   (b) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the injectivity of f. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   > [We want to verify the statement '*for any $x, w \in \mathbb{R}$, if $f(x) = f(w)$ then $x = w$.*']
   >
   > Pick any _____(I)_____ .
   > _____(II)_____ $f(x) = f(w)$.
   > Then $x^{\frac{5}{3}} =$ _____(III)_____ $= w^{\frac{5}{3}}$.
   > Since $x, w \in \mathbb{R}$, we have $x = (x^{\frac{5}{3}})^{\frac{3}{5}} =$ _____(IV)_____ .
   > It follows that _____(V)_____ .

3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{x}{x^2+1}$ for any $x \in \mathbb{R}$.

   (a) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the non-surjectivity of f. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

[We want to verify the statement '*there exists some* $y_0 \in \mathbb{R}$ *such that for any* $x \in \mathbb{R}$ *such that* $y \neq f(x)$.']

$\underline{\quad\text{(I)}\quad}$ $y_0 = 1$.

We verify, using the method of proof-by-contradiction, that $\underline{\qquad\text{(II)}\qquad}$, $f(x) \neq y_0$:

- $\underline{\quad\text{(III)}\quad}$ it were true that $\underline{\qquad\text{(IV)}\qquad}$.

  Then $\dfrac{x_0}{x_0{}^2+1} = f(x_0) = y_0 = 1$.

  Therefore $\underline{\quad\text{(V)}\quad}$ $= x_0{}^2 - x_0 + 1 =$ $\underline{\qquad\text{(VI)}\qquad}$ > 0. Contradiction arises.

It follows that $\underline{\qquad\text{(VII)}\qquad}$.

(b) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the non-injectivity of f. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

[We want to verify the statement '*there exists some* $x_0, w_0 \in \mathbb{R}$, *such that* $f(x_0) = f(w_0)$ *and* $x \neq w$.']

Take $x_0 = \dfrac{1}{2}$, $\underline{\quad\text{(I)}\quad}$.

Note that $x_0, w_0 \in \mathbb{R}$.

Also note that $\underline{\qquad\text{(II)}\qquad}$.

We have $f(x_0) = \dfrac{1/2}{(1/2)^2+1} = \dfrac{2}{5}$ and $\underline{\qquad\text{(III)}\qquad}$.

Then $f(x_0) = \underline{\quad\text{(IV)}\quad}$.

It follows that $\underline{\qquad\text{(V)}\qquad}$.

4. Let $f : (0, +\infty) \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{x^2-2x+4}{x^2+2x+4}\cos\left(\dfrac{1}{x\sqrt{x}}\right)$ for any $x \in (0, +\infty)$.

   (a) Verify that f is not injective.

   (b) i. Verify that $\left|\dfrac{x^2-2x+4}{x^2+2x+4}\right| \leq 1$ for any $x \in (0, +\infty)$.

   Remark. A very simple answer can be obtained without using calculus.

   ii. Apply the previous part, or otherwise, to verify that f is not surjective.

5.$^\diamond$ Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{Q} \\ x^3 & \text{if} \quad x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$.

   (a) Is f injective? Justify your answer.

   (b) Is f surjective? Justify your answer.

6. Let $n \in \mathbb{N}\backslash\{0, 1\}$, and $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = z^n$ for any $z \in \mathbb{C}$.

   (a) Verify that f is surjective.

   (b) Is f injective? Why?

7. Let $f : \mathbb{C}\backslash\{0\} \longrightarrow \mathbb{C}\backslash\{0\}$ be the function defined by $f(z) = \dfrac{z}{\bar{z}}$ for any $z \in \mathbb{C}\backslash\{0\}$.

   (a) Is f injective? Why?

   (b) Is f surjective? Why?

8. We introduce the definition below:

   - *Let D be a subset of $\mathbb{C}$, and $f : D \longrightarrow \mathbb{C}$ be a function. Let $\zeta \in D$. ζ is said to be a* **zero of f in D** *if $f(\zeta) = 0$.*

In this question, you are supposed to be familiar with the notion of continuity in the calculus of one real variable. You may take for granted the validity of **Bolzano's Intermediate Value Theorem***:*

- *Let $a, b \in \mathbb{R}$, with $a < b$, and $f : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose f is continuous on $[a, b]$. Further suppose $f(a)f(b) < 0$. Then f has a zero in (a, b).*

(a)♣ Let $p, q, r \in \mathbb{R}$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^3 + px^2 + qx + r$ for any $x \in \mathbb{R}$.
You may take for granted that f is continuous on $\mathbb{R}$.
Define $b = 1 + 2(|p| + |q| + |r|)$, and $a = -b$.

i. Prove that $f(b) \geq \frac{b^3}{2}$ and $f(a) \leq -\frac{b^3}{2}$.

ii. Hence apply Bolzano's Intermediate Value Theorem to deduce that f has a zero in (a, b).

(b)♢ Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a cubic polynomial function with real coefficients. Prove that g is surjective.

9. Let A be the subset of $\mathbb{N}$ defined by $A = \{0, 1, 2, 3, 4, 5\}$, and $f : A \longrightarrow A$ be the function defined by $f(0) = 1$, $f(1) = 1$, $f(2) = 2$, $f(3) = 2$, $f(4) = 2$, $f(5) = 5$.

Consider each of the sets below. Where it is not the empty set, list every element of the set concerned, each element exactly once. Where it is the empty set, write '*it is the empty set*'.

(a) $f(\{1, 2, 3\})$
(b) $f^{-1}(\{1\})$
(c) $f^{-1}(\{3, 4\})$
(d) $f(\{0, 2, 4\})$
(e) $f^{-1}(\{0, 2, 4\})$
(f) $f(f^{-1}(\{0, 2, 4\}))$
(g) $f^{-1}(f(\{0, 2, 4\}))$
(h) $(f \circ f)(\{0, 2, 4\})$
(i) $(f \circ f)^{-1}(\{0, 2, 4\})$

10. Let $f : \mathbb{R}\backslash\{0, 2\} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \frac{2}{x(x-2)} + 1$ for any $x \in \mathbb{R}\backslash\{0, 2\}$.

Write down the respective values of the numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \kappa, \lambda, \mu, \nu, \xi, \rho, \sigma, \tau, \varphi, \psi, \omega$, so that the set equalities below hold. You are not required to justify your answer.

(a) $f((0, 2)) = (-\infty, \alpha]$.
(b) $f([4, +\infty)) = (\beta, \gamma]$.
(c) $f((1, 3)\backslash\{2\}) = (-\infty, \delta) \cup (\varepsilon, +\infty)$.
(d) $f^{-1}(\{3\}) = \{\zeta, \eta\}$.
(e) $f^{-1}([3, 9]) = [\theta, \kappa] \cup [\lambda, \mu]$.
(f) $f^{-1}([-3, 0]) = [\nu, \xi]$.
(g) $f^{-1}([-3, 1]) = [\rho, \sigma]$.
(h) $f^{-1}([-3, 3]) = (-\infty, \tau] \cup [\varphi, \psi] \cup [\omega, +\infty)$.

11. Define the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x^{-2} & \text{if} \quad x < -1 \\ -1 & \text{if} \quad x = -1 \\ -x & \text{if} \quad -1 < x < 0 \\ 2 & \text{if} \quad x = 0 \\ 2x^2 + 1 & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x = 1 \\ 1 + x^{-1} & \text{if} \quad x > 1 \end{cases}.$$

Write down the respective values of the numbers α, β, γ, δ, ε, ζ, η, θ, κ, so that the set equalities below hold. You are not required to justify your answer. (But it may help if you draw the graph of f first.)

(a) $f(\mathbb{R}) = ([\alpha, \beta)\backslash\{\gamma\}) \cup \{\delta\}$.
(b) $f^{-1}([0.25, 3]) = ([\varepsilon, \zeta]\backslash\{\eta\}) \cup ([\theta, +\infty)\backslash\{\kappa\})$.

12. *You are not required to justify your answers in this question.*

Let $a, b \in \mathbb{R}$, and $f : [0, 5] \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} -\frac{12}{(x+1)(x-3)} & \text{if} \quad 0 \leq x < 3 \\ a & \text{if} \quad x = 3 \\ -(x-3)(x-5) & \text{if} \quad 3 < x \leq 5 \text{ and } x \neq 4 \\ b & \text{if} \quad x = 4 \end{cases}.$$

Suppose $f(3) < f(4)$. Further suppose that $f^{-1}(\{2\}) \neq \emptyset$ and $f^{-1}(\{3\})$ has exactly two elements.

(a) What are the respective values of a, b?

(b) Name all two elements of $f^{-1}(\{3\})$.

(c) What are the numbers $\alpha, \beta, \gamma, \delta$ for which the set equality $f([2,4)) = (\alpha, \beta) \cup \{\gamma\} \cup [\delta, +\infty)$ holds?