MATH1050 Exercise 9, Exercise 10 Supplement

1. Denote the interval $(0, +\infty)$ by $I$. Let $f : I \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \frac{1}{2}\left(x - \frac{1}{x}\right)$ for any $x \in I$.
   (a) Verify that $f$ is injective directly from the definition of injectivity.
   (b) Verify that $f$ is surjective directly from the definition of surjectivity.
   **Remark.** It may help if you start by considering whether, for each $b \in \mathbb{R}$, the equation $b = f(u)$ with unknown $u$ has any solution or not.

2. Let $f : [0, 9] \longrightarrow \mathbb{R}$ be the function defined by $f(x) = -x + 6\sqrt{x} - 5$ for any $x \in [0, 9]$.
   (a) Show that $f(x) = -(A - \sqrt{x})^2 + B$ for any $x \in [0, 9]$. Here $A, B$ are some real constants whose respective values you have to determine.
   (b) Verify that $f$ is injective directly from the definition of injectivity.
   (c) Is $f$ surjective? Justify your answer directly from the definition of surjectivity.

3. Let $f : (0, +\infty) \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \frac{x^2 - 1}{x^2 + 1} \sin\left(\frac{1}{\sqrt{x}}\right)$ for any $x \in (0, +\infty)$.
   (a) Verify that $f$ is not injective.
   (b) i. Verify that $\left|\frac{x^2 - 1}{x^2 + 1}\right| \leq 1$ for any $x \in (0, +\infty)$.
   **Remark.** A very simple answer can be obtained without using calculus.
   ii. Apply the previous part, or otherwise, to verify that $f$ is not surjective.

4.$^\diamond$ Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \begin{cases} 2x + 5 & \text{if} \quad x \leq -3 \\ -x & \text{if} \quad -3 < x < 1 \\ x - 3 & \text{if} \quad x \geq 1 \end{cases}$.
   (a) Is $f$ surjective? Why?
   (b) Is $f$ injective? Why?

5. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \begin{cases} x & \text{if} \quad x \geq 0 \\ -2x & \text{if} \quad x < 0 \end{cases}$.
   (a) Is $f$ surjective? Why?
   (b) Is $f$ injective? Why?

6.$^\diamond$ Let $f : (0, +\infty) \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \begin{cases} x + 2 & \text{if} \quad x > 0 \text{ and } x \text{ is rational} \\ 2x - 1 & \text{if} \quad x > 0 \text{ and } x \text{ is irrational} \end{cases}$.
   (a) Is $f$ surjective? Why?
   (b) Is $f$ injective? Why?

7.$^\diamond$ Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{Q} \\ x^2 & \text{if} \quad x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$.
   (a) Is $f$ injective? Justify your answer.
   (b) Is $f$ surjective? Justify your answer.

8.$^\diamond$ Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be the functions respectively defined by $f(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ x^3 & \text{if} \quad x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$, $g(x) = \begin{cases} x^3 & \text{if} \quad x \in \mathbb{Q} \\ x & \text{if} \quad x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$.
   (a) i. Is $f$ injective? Justify your answer.
   ii. Is $f$ surjective? Justify your answer.
   (b) i. Is $g$ injective? Justify your answer.
   ii. Is $g$ surjective? Justify your answer.

9. (a) Let $A = (0, +\infty)$, and $f : A \longrightarrow A$ be the function defined by $f(x) = \frac{1}{x^2}$ for any $x \in A$.
   i. Verify that $f$ is surjective.
   ii. Verify that $f$ is injective.

(b)$^\diamondsuit$ Let $B = (0, +\infty) \cap \mathbb{Q}$, and $g : B \longrightarrow B$ be the function defined by $g(x) = \dfrac{1}{x^2}$ for any $x \in B$.

i. Verify that $g$ is not surjective.
ii. Is $g$ injective? Why?

10. (a) Let $f : \mathbb{R}\backslash\{-1, 1\} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{1}{x^2 - 1}$ for any $x \in \mathbb{R}\backslash\{-1, 1\}$.

i. Verify that $f$ is not injective.
ii. Verify that $f$ is not surjective.

(b) i. Let $x \in \mathbb{R}\backslash\{-1, 1\}$. Verify that $\dfrac{1}{1 - x^2} \in \mathbb{R}\backslash(-1, 0]$.

ii. Let $y \in \mathbb{R}\backslash(-1, 0]$. Verify that there exists some $x \in \mathbb{R}\backslash\{-1, 1\}$ such that $y = \dfrac{1}{1 - x^2}$.

iii. Let $g : \mathbb{R}\backslash\{-1, 1\} \longrightarrow \mathbb{R}\backslash(-1, 0]$ be the function defined by $g(x) = \dfrac{1}{x^2 - 1}$ for any $x \in \mathbb{R}\backslash\{-1, 1\}$.
Is $g$ surjective? Why?

**Remark.** What is the point of this question? Starting from a non-surjective function, if we 'restrict' its range appropriately, we will obtain a new function which is surjective. However, we need be careful not to 'over-restrict' it.

11. (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^2 - 2x$ for any $x \in \mathbb{R}$.

i. Is $f$ injective? Justify your answer.
ii. Is $f$ surjective? Justify your answer.

(b) Verify that for any $x \in (1, +\infty)$, $x^2 - 2x > -1$.

(c) Let $g : (1, +\infty) \longrightarrow (-1, +\infty)$ be the function defined by $g(x) = x^2 - 2x$ for any $x \in (1, +\infty)$.

i. Is $g$ injective? Justify your answer.
ii. Is $g$ surjective? Justify your answer.
iii. Is $g$ bijective? If *yes*, also write down the 'formula of definition' for its inverse function.

12. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $g(x) = \dfrac{10^x - 10^{-x}}{2}$ for any $x \in \mathbb{R}$.

(a) Verify that $g$ is injective.
(b) Verify that $g$ is surjective.
(c) What is the 'formula of definition' of the function $g^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$?

**Remark.** There is no need to use any results from the calculus in this question.

13. (a) Prove that for any $t \in \mathbb{R}$, $0 < \dfrac{1}{\sqrt{1 + e^{-t}}} < 1$.

(b) Denote the interval $(0, 1)$ by $I$. Define the function $g : \mathbb{R} \longrightarrow I$ by $g(x) = \dfrac{1}{\sqrt{1 + e^{-x}}}$ for any $x \in \mathbb{R}$.

i. Verify that $g$ is surjective, directly from the definition of surjectivity.
ii. Verify that $g$ is injective, directly from the definition of injectivity.
iii. Is $g$ bijective? If *yes*, also write down the 'formula of definition' for its inverse function.

14. (a) Prove that for any $x \in \mathbb{R}\backslash\{2\}$, $\dfrac{3x}{x - 2} \neq 3$.

(b) Let $f : \mathbb{R}\backslash\{2\} \longrightarrow \mathbb{R}\backslash\{3\}$ be the function defined by $f(x) = \dfrac{3x}{x - 2}$ for any $x \in \mathbb{R}\backslash\{2\}$.

i. Is $f$ injective? Justify your answer.
ii. Is $f$ surjective? Justify your answer.
iii. Is $f$ bijective? If *yes*, also write down the 'formula of definition' for its inverse function.

15. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x|x|$ for any $x \in \mathbb{R}$.

(a) Is $f$ injective? Justify your answer.
(b) Is $f$ surjective? Justify your answer.
(c) Is $f$ bijective? If *yes*, also write down the 'formula of definition' for its inverse function.

16. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x\sqrt{|x|}$ for any $x \in \mathbb{R}$.

(a) Is $f$ injective? Justify your answer.

(b) Is $f$ surjective? Justify your answer.

(c) Is $f$ bijective? If *yes*, also write down the 'formula of definition' for its inverse function.

17. (a) Prove that for any $x \in \mathbb{R}$, $-1 < \dfrac{x|x|}{x^2+1} < 1$.

(b) Let $f : \mathbb{R} \longrightarrow (-1,1)$ be the function defined by $f(x) = \dfrac{x|x|}{x^2+1}$ for any $x \in \mathbb{R}$.

i. Is $f$ injective? Justify your answer.

ii.$^\diamondsuit$ Is $f$ surjective? Justify your answer.

iii. Is $f$ bijective? If *yes*, also write down the 'formula of definition' for its inverse function.

18. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = z^2 - 6z + 13$ for any $z \in \mathbb{C}$.

(a)$^\diamondsuit$ Verify that $f$ is surjective.

(b) Verify that $f$ is not injective.

19. Let $f : \mathbb{C}\backslash\{0\} \longrightarrow \mathbb{C}\backslash\{0\}$ be the function defined by $f(z) = \dfrac{z^2}{\bar{z}}$ for any $z \in \mathbb{C}\backslash\{0\}$.

(a) Verify that $f(z) = \dfrac{z^3}{|z|^2}$ for any $z \in \mathbb{C}\backslash\{0\}$.

(b)$^\diamondsuit$ Is $f$ injective? Justify your answer.

(c)$^\clubsuit$ Is $f$ surjective? Justify your answer.

20. (a) Let $f : \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x + \dfrac{1}{x}$ for any $x \in \mathbb{R}\backslash\{0\}$.

i.$^\diamondsuit$ Verify that $f$ is not surjective.

ii.$^\diamondsuit$ Verify that $f$ is not injective.

(b) Verify that for any $x \in \mathbb{R}\backslash\{0\}$, $\left|x + \dfrac{1}{x}\right| > 2$.

(c) Let $g : \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}\backslash(-2,2)$ be the function defined by $g(x) = x + \dfrac{1}{x}$ for any $x \in \mathbb{R}\backslash\{0\}$.

i.$^\clubsuit$ Verify that $g$ is surjective.

ii.$^\diamondsuit$ Is $g$ injective? Why?

(d) Let $h : \mathbb{C}\backslash\{0\} \longrightarrow \mathbb{C}$ be the function defined by $h(z) = z + \dfrac{1}{z}$ for any $z \in \mathbb{C}\backslash\{0\}$.

i.$^\clubsuit$ Verify that $h$ is surjective.

ii.$^\diamondsuit$ Is $h$ injective? Why?

21.$^\clubsuit$ Let $A = \{z \in \mathbb{C} : -\pi < \mathsf{Im}(z) < \pi\}$, $L = \{w \in \mathbb{C} : \mathsf{Im}(w) = 0 \text{ and } \mathsf{Re}(w) \leq 0\}$, and $B = \mathbb{C}\backslash L$.

(a) Prove that for any $z \in A$, $e^{\mathsf{Re}(z)}(\cos(\mathsf{Im}(z)) + i\sin(\mathsf{Im}(z))) \in B$.

(b) Define the function $f : A \longrightarrow B$ by $f(z) = e^{\mathsf{Re}(z)}(\cos(\mathsf{Im}(z)) + i\sin(\mathsf{Im}(z)))$ for any $z \in A$.

i. Verify that $f$ is injective.

ii. Verify that $f$ is surjective.

22. (a) Prove that for any $z \in \mathbb{C}\backslash\{-2\}$, $\dfrac{z+i}{z+2} \neq 1$.

(b) Let $f : \mathbb{C}\backslash\{-2\} \longrightarrow \mathbb{C}\backslash\{1\}$ be the function defined by $f(z) = \dfrac{z+i}{z+2}$ for any $z \in \mathbb{C}\backslash\{-2\}$.

i. Prove that $f$ is injective.

ii. Prove that $f$ is surjecive.

iii. What is the 'formula of definition' of the inverse function of $f$, namely, the function $f^{-1} : \mathbb{C}\backslash\{1\} \longrightarrow \mathbb{C}\backslash\{-2\}$?

23. For any $a, b \in \mathbb{C}$, define the function $f_{a,b} : \mathbb{C} \longrightarrow \mathbb{C}$ by $f_{a,b}(z) = az + b\bar{z}$ for any $z \in \mathbb{C}$.

(a) Let $a, b, c, d \in \mathbb{C}$.

i. Verify that $f_{c,d}(f_{a,b}(z)) = (ac + \bar{b}d)z + (bd + \bar{a}d)\bar{z}$ for any $z \in \mathbb{C}$.

ii. Verify that $f_{ac,bc}(z) = cf_{a,b}(z)$ for any $z \in \mathbb{C}$.

iii. Suppose $c \in \mathbb{R}$. Verify that $f_{ac,bc}(z) = f_{a,b}(cz)$ for any $z \in \mathbb{C}$.

(b)$^\clubsuit$ Let $a, b \in \mathbb{C}$. Prove that there exist some $\alpha, \beta \in \mathbb{C}$ such that $f_{\alpha,\beta}(f_{a,b}(z)) = (|a|^2 - |b|^2)z$ for any $z \in \mathbb{C}$.

**Remark**. Make use of part (a.i) to find candidates for $\alpha, \beta$.

(c)$^\clubsuit$ Let $a, b \in \mathbb{C}$. Suppose $|a| \neq |b|$. Prove that $f_{a,b}$ is bijective. What is the 'formula of definition' of the inverse function of $f_{a,b}$?

**Remark**. Instead of checking surjectivity and injectivity directly from definition, make use of parts (b), (a.ii), (a.iii) to write down a candidate inverse function for the function $f_{a,b}$.

(d)$^\clubsuit$ Let $a, b \in \mathbb{C}$. Suppose $|a| = |b|$. Is $f_{a,b}$ bijective? Justify your answer.

24. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = z(\mathsf{Re}(z))^2 + i\mathsf{Im}(z)$ for any $z \in \mathbb{C}$.

(a) Verify that $\begin{cases} \mathsf{Re}(f(z)) &= A(\mathsf{Re}(z))^M \\ \mathsf{Im}(f(z)) &= [B(\mathsf{Re}(z))^N + C] \cdot \mathsf{Im}(z) \end{cases}$ for any $z \in \mathbb{C}$.

Here $A, B, C, M, N$ are integers whose values you have to determine explicitly.

(b) Verify that $f$ is injective, directly from the definition of injectivity.

(c)$^\diamondsuit$ Verify that $f$ is surjective, directly from the definition of surjectivity.

(d) Write down the 'formula of definition' of the inverse function $f^{-1} : \mathbb{C} \longrightarrow \mathbb{C}$ of the function $f$.

25. (a)$^\diamondsuit$ Let $n \in \mathbb{N}\backslash\{0\}$, and $a \in \mathbb{C}\backslash\{0\}$. Define the function $\mu : \mathbb{C} \longrightarrow \mathbb{C}$ by $\mu(z) = az^n$ for any $z \in \mathbb{C}$.

Prove that $\mu$ is bijective iff $n = 1$.

(b) Let $h : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by

$$h(z) = \begin{cases} iz & \text{if} \quad |z| \in \mathbb{Q} \\ \dfrac{3i}{2\bar{z}} & \text{if} \quad |z| \in \mathbb{R}\backslash\mathbb{Q} \end{cases}.$$

i. Prove the statement below:

- *For any $\zeta \in \mathbb{C}$, if $|\zeta|$ is irrational then $|h(\zeta)|$ is irrational.*

ii.$^\diamondsuit$ Prove that $h \circ h$ is a polynomial function on $\mathbb{C}$. Determine the explicit formula of definition for $h \circ h$ as well.

iii.$^\diamondsuit$ Is $h$ bijective? Justify your answer. (*Hint.* Make good use of the result in the previous part.)

26.$^\clubsuit$ We introduce this definition:

*A* **monic polynomial** *is a polynomial whose leading coefficient is* 1.

Denote by $M$ the set of all monic quadratic polynomials with real coefficients and indeterminate $x$.

(a) For any $\alpha, \beta \in \mathbb{R}$, denote by $p_{\alpha,\beta}(x)$ the monic quadratic polynomial $(x - \alpha)(x - \beta)$ with indeterminate $x$.

Define the function $\Phi : \mathbb{R}^2 \longrightarrow M$ by $(\alpha, \beta) \underset{\Phi}{\longmapsto} p_{\alpha,\beta}(x)$ for any $\alpha, \beta \in \mathbb{R}$.

i. Is $\Phi$ surjective? Justify your answer.

ii. Is $\Phi$ injective? Justify your answer.

(b) For any $\alpha, \beta \in \mathbb{R}$, denote by $q_{\alpha,\beta}(x)$ the monic quadratic polynomial $(x - \alpha)^2 + \beta$ with indeterminate $x$.

Define the function $\Psi : \mathbb{R}^2 \longrightarrow M$ by $(\alpha, \beta) \underset{\Psi}{\longmapsto} q_{\alpha,\beta}(x)$ for any $\alpha, \beta \in \mathbb{R}$.

i. Is $\Psi$ surjective? Justify your answer.

ii. Is $\Psi$ injective? Justify your answer.

27. (a) Prove each of the statements below:

i. *Let $A, B, C$ be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose $g \circ f$ is surjective. Then $g$ is surjective.*

ii. *Let $A, B, C$ be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose $g \circ f$ is injective. Then $f$ is injective.*

(b)$^\clubsuit$ Let $I, J, K$ be sets, and $\alpha : I \longrightarrow J$, $\beta : J \longrightarrow K$, $\gamma : K \longrightarrow I$ be functions. Suppose $\gamma \circ \beta \circ \alpha$, $\alpha \circ \gamma \circ \beta$ are both injective. Further suppose $\beta \circ \alpha \circ \gamma$ is surjective.

Prove that each of the functions $\alpha, \beta, \gamma$ is both surjective and injective.

28. Consider each of the statements below. Dis-prove it with an appropriate argument.

(a) *Let $A, B, C$ be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose $g \circ f$ is surjective. Then $f$ is surjective.*

(b) *Let $A, B, C$ be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose $g \circ f$ is injective. Then $g$ is injective.*

29. Consider each of the statements below. For each of them, dis-prove it by constructing an appropriate counter-example.

(a) *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions. Suppose $f, g$ are injective. Then the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = f(x) + g(x)$ for any $x \in \mathbb{R}$ is injective.*

(b) *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions. Suppose $f, g$ are injective. Then the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = f(x) + g(x)$ for any $x \in \mathbb{R}$ is not injective.*

(c) *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions. Suppose $f, g$ are injective. Then the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = f(x)g(x)$ for any $x \in \mathbb{R}$ is injective.*

(d) *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions. Suppose $f, g$ are injective. Then the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = f(x)g(x)$ for any $x \in \mathbb{R}$ is not injective.*

(e) *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions. Suppose $f, g$ are surjective. Then the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = f(x) + g(x)$ for any $x \in \mathbb{R}$ is surjective.*

(f) *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions. Suppose $f, g$ are surjective. Then the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = f(x) + g(x)$ for any $x \in \mathbb{R}$ is not surjective.*

(g) *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions. Suppose $f, g$ are surjective. Then the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = f(x)g(x)$ for any $x \in \mathbb{R}$ is surjective.*

(h) *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions. Suppose $f, g$ are surjective. Then the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = f(x)g(x)$ for any $x \in \mathbb{R}$ is not surjective.*

30.$^\diamondsuit$ We introduce this definition below:

- *Let $A, B, C, D$ be sets, and $f : A \longrightarrow C$, $g : B \longrightarrow D$ be functions. Suppose $f(x) = g(x)$ for any $x \in A \cap B$. Define the function $f \cup g : A \cup B \longrightarrow C \cup D$ by $(f \cup g)(x) = \begin{cases} f(x) & \text{if} \quad x \in A \\ g(x) & \text{if} \quad x \in B \end{cases}$.*

  *The function $f \cup g$ is called the* **union of the functions** *$f, g$.*

Consider each of the statements below. Determine whether it is true or false. Justify your answer with an appropriate argument.

(a) *Let $A, B, C, D$ be sets, and $f : A \longrightarrow C$, $g : B \longrightarrow D$ be functions. Suppose $f(x) = g(x)$ for any $x \in A \cap B$. Suppose $f, g$ are surjective. Then $f \cup g : A \cup B \longrightarrow C \cup D$ is surjective.*

(b) *Let $A, B, C, D$ be sets, and $f : A \longrightarrow C$, $g : B \longrightarrow D$ be functions. Suppose $f(x) = g(x)$ for any $x \in A \cap B$. Suppose $f, g$ are injective. Then $f \cup g : A \cup B \longrightarrow C \cup D$ is injective.*

31.$^\heartsuit$ *You need recall the Fundamental Theorem of the Calculus (and other things in the calculus of one real variable).*

Let $J$ be an open interval in $\mathbb{R}$. (It is assumed that $J$ is not the empty set.) Denote by $C(J)$ the set of all real-valued continuous functions on $J$. Denote by $C^1(J)$ the set of all real-valued differentiable functions on $J$ whose first derivatives are continuous functions on $J$.

Define the function $D : C^1(J) \longrightarrow C(J)$ by $D(\varphi) = \varphi'$ for any $\varphi \in C^1(J)$.

For each $a \in J$, define the function $I_a : C(J) \longrightarrow C^1(J)$ by $(I_a(\psi))(x) = \displaystyle\int_a^x \psi(t)dt$ for any $\psi \in C(J)$ for any $x \in J$.

(a) i. Is the function $D : C^1(J) \longrightarrow C(J)$ surjective?
   ii. Is the function $D : C^1(J) \longrightarrow C(J)$ injective?

(b) Let $a \in J$.
   i. Prove that $((I_a \circ D)(\varphi))(x) = \varphi(x) - \varphi(a)$ for any $\varphi \in C^1(J)$ for any $x \in J$.
   ii. Prove that $((D \circ I_a)(\psi))(x) = \psi(x)$ for any $\psi \in C(J)$ for any $x \in J$.

(c) Let $a \in J$.
   i. Is the function $I_a : C(J) \longrightarrow C^1(J)$ surjective?
   ii. Is the function $I_a : C(J) \longrightarrow C^1(J)$ injective?
   iii. Is the function $I_a \circ D : C^1(J) \longrightarrow C^1(J)$ surjective?
   iv. Is the function $I_a \circ D : C^1(J) \longrightarrow C^1(J)$ injective?

32.$^\heartsuit$ *Familiarity with the calculus of one real variable is assumed in this question.*

We recall this definition from the calculus of one real variable:

- *A real-valued function of one real variable is said to be* **smooth** *if it is differentiable for as many times as we like at every point of its domain.*

Denote by $C^\infty(\mathbb{R})$ the set of all smooth functions on $\mathbb{R}$.

Let $X : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ be the function defined by $(X(\varphi))(x) = x\varphi(x)$ for any $\varphi \in C^\infty(\mathbb{R})$ for any $x \in \mathbb{R}$.

Let $D : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ be the function defined by $(D(\varphi))(x) = \varphi'(x)$ for any $\varphi \in C^\infty(\mathbb{R})$ for any $x \in \mathbb{R}$.

Let $I_0 : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ be the function defined by $(I_0(\varphi))(x) = \displaystyle\int_0^x \varphi(t)dt$ for any $\varphi \in C^\infty(\mathbb{R})$ for any $x \in \mathbb{R}$.

(a) Verify that for any $\varphi \in C^\infty(\mathbb{R})$, for any $x \in \mathbb{R}$, $((D \circ X)(\varphi))(x) - (X \circ D)(\varphi))(x) = \varphi(x)$.

**Remark.** This seemingly innocuous mathematical statement is a baby case of something of great significance in *modern physics*; it is behind the **Heisenberg relations** for position and momentum in *quantum mechanics*.

(b) Verify that for any $\varphi \in C^\infty(\mathbb{R})$, for any $x \in \mathbb{R}$, $((X \circ I_0)(\varphi))(x) - (I_0 \circ X)(\varphi))(x) = ((I_0 \circ I_0)(\varphi))(x)$.

33. *In this question, you are supposed to be familiar with the notion of continuity in the calculus of one real variable. You may take for granted the validity of* **Bolzano's Intermediate Value Theorem**.

- *Let $a, b \in \mathbb{R}$, with $a < b$, and $f : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose $f$ is continuous on $[a, b]$. Further suppose $f(a)f(b) < 0$. Then $f$ has a zero in $(a, b)$.*

(a)♣ Let $p, q, r, s, t \in \mathbb{R}$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^5 + px^4 + qx^3 + rx^2 + sx + t$ for any $x \in \mathbb{R}$.

*You may take for granted that $f$ is continuous on $\mathbb{R}$.*

Define $b = 1 + 2(|p| + |q| + |r| + |s| + |t|)$, and $a = -b$.

i. Prove that $f(b) \geq \dfrac{b^5}{2}$ and $f(a) \leq -\dfrac{b^5}{2}$.

ii. Hence apply Bolzano's Intermediate Value Theorem to deduce that $f$ has a zero in $(a, b)$.

(b)♢ Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a quintic polynomial function with real coefficients. Prove that $g$ is surjective.

**Remark.** Can you imitate the argument above to prove that every polynomial function with real coefficient of odd degree is a surjective function from $\mathbb{R}$ to $\mathbb{R}$? What fails to work if you try such an argument on a polynomial function with real coefficient of even degree?

34. *In this question, we are illustrating via specific examples the Cardano-Tartaglia method for finding roots of a general cubic polynomial with complex coefficients.*

Let $\omega$ be a cube root of unity. Suppose $\omega \neq 1$.

(a) Verify the 'identities' below, in which each of $a, b, c$ may stand for a complex number or an indeterminate:

i. $a^2 + b^2 + c^2 - ab - bc - ca = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$.

ii. $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$.

(b) Consider the polynomial $f(x) = x^3 - 9x + 12$, in which $x$ is the indeterminate.

i. Find real numbers $\kappa, \lambda$ which satisfy $\kappa \leq \lambda$ and $\kappa^3 + \lambda^3 = 12$ and $\kappa\lambda = 3$.

ii.♢ Hence, by factorizing $f(x)$, or otherwise, find the roots of $f(x)$ in terms of $\omega$.

**Remark.** First re-express $f(x)$ in such a way that $\kappa^3, \lambda^3, \kappa\lambda$ appear explicitly.

(c) Consider the polynomial $g(y) = y^3 + 3y^2 - 12y + 10\sqrt{5} - 14$, in which $y$ is the indeterminate.

i. With an appropriate substitution $y = x - \alpha$, in which $\alpha$ is an appropriate constant, re-express $g(y)$ as $x^3 + sx + t$ in which $s, t$ are constants.

ii.♢ Hence find the roots of $g(y)$.

35. Let $\omega = \cos\left(\dfrac{2\pi}{3}\right) + i\sin\left(\dfrac{2\pi}{3}\right)$. Note that $\omega$ is one of the two cube roots of unity which are not 1.

(a) For the moment, take for granted the validity of the statement (♯) below:

(♯) *Let $h, k \in \mathbb{C}$. There exist some $\sigma, \tau \in \mathbb{C}$ such that $k = \sigma^3 + \tau^3$ and $h = -3\sigma\tau$.*

Let $s, t \in \mathbb{C}$. Consider the polynomial $f(x) = x^3 + sx + t$ with indeterminant $x$. By factorizing $f(x)$, prove that the polynomial $f(x)$ has a root in $\mathbb{C}$.

**Remark.** You may need the 'identity' $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$.

(b) Let $p, q, r \in \mathbb{C}$. Consider the polynomial $g(y) = y^3 + py^2 + qy + r$ with indeterminate $y$.

i. Prove that there exists some $\alpha, s, t \in \mathbb{C}$ such that $g(y) = (y - \alpha)^3 + s(y - \alpha) + t$ as polynomials.

ii. Apply the result in the previous part to prove that the polynomial $g(y)$ has a root in $\mathbb{C}$.

(c)♢ Let $G : \mathbb{C} \longrightarrow \mathbb{C}$ be a cubic polynomial function with complex coefficients. Prove that $G$ is surjective.

(d)$^\diamondsuit$ *We are going to prove the statement* ($\sharp$) *here.*

Let $h, k \in \mathbb{C}$. Consider the quadratic polynomial $Q(u) = u^2 - ku - \dfrac{h^3}{27}$ with indeterminate $u$. Take for granted that the roots of $Q(u)$ in $\mathbb{C}$ are $\mu, \nu$ respectively, and $Q(u) = (u - \mu)(u - \nu)$ as polynomials.

i. Verify that $\mu + \nu = k$ and $\mu\nu = -\dfrac{h^3}{27}$.

ii. Hence prove that there exist some $\sigma, \tau \in \mathbb{C}$ such that $\sigma^3 + \tau^3 = k$ and $\sigma\tau = -\dfrac{h}{3}$.

**Remark.** Combined together, the results described in this question tell us how we may solve arbitrary cubic polynomial equations with complex coefficients, with the help of the operations $+, -, \times, \div$ and 'taking square roots', 'taking cube roots'. This is the **Cardano-Tartaglia Method**. (But what about quartic polynomial equations? Quintic polynomial equations? You will know the answer from your *algebra* courses.)

36. We introduce/recall these definitions:

- *Let $n \in \mathbb{N}$. A* **degree-$n$ polynomial with complex coefficients and with indeterminate** *$z$ is an expression of the form $a_n z^n + \cdots + a_1 z + a_0$ in which $a_0, a_1, \cdots, a_n \in \mathbb{C}$ and $a_n \neq 0$.*
- *A complex-valued function of one complex variable is called a* **degree-$n$ polynomial function on $\mathbb{C}$** *exactly when its 'formula of definition' is given by a degree-$n$ polynomial with complex coefficients.*
- *Let $\zeta \in \mathbb{C}$, and $f(z) \equiv a_n z^n + \cdots + a_1 z + a_0$ be a polynomial with complex coefficients and with indeterminate $z$. $\zeta$ is said to be a root of the polynomial $f(z)$ in $\mathbb{C}$ if $f(\zeta) = 0$.*

The statement ($\sharp$) below, first proved by Gauss, is known as the **Fundamental Theorem of Algebra**:

($\sharp$) *Every non-constant polynomial with complex coefficients (and with one indeterminate) has a root in $\mathbb{C}$.*

(a)$^\diamondsuit$ Prove that the statement ($\sharp$) is logically equivalent to the statement ($\flat$) below:

($\flat$) *For any $n \in \mathbb{N}\backslash\{0\}$, every degree-$n$ polynomial function on $\mathbb{C}$ is surjective.*

(b)$^\clubsuit$ Let $n \in \mathbb{N}\backslash\{0, 1\}$, $a_0, a_1, \cdots, a_n \in \mathbb{C}$, with $a_n \neq 0$, and $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the degree-$n$ polynomial function defined by $f(z) = a_n z^n + \cdots + a_1 z + a_0$ for any $z \in \mathbb{C}$.
Apply the Fundamental Theorem of Algebra, or otherwise, to prove that $f$ is not injective.
**Remark.** Here you may also take for granted the **Factor Theorem** (whose 'real version' you have already learnt at school and may be carried in verbatim to the 'complex situation'):

- *Let $\alpha \in \mathbb{C}$, and $p(z)$ be a degree-$n$ polynomial (with complex coefficients). Suppose $\alpha$ is a root of $p(z)$. Then there is a degree-$(n-1)$ polynomial $q(z)$ (with complex coefficients) so that $p(z) = (z - \alpha)q(z)$ as polynomials.*

37. Let $A = \{0, 1, 2, 3, 4\}$, $B = \{5, 6, 7, 8, 9\}$. Define the function $f : A \longrightarrow B$ by $f(0) = 5$, $f(1) = 5$, $f(2) = 7$, $f(3) = 9$, $f(4) = 9$.

Consider each of the sets below. Where it is not the empty set, list every element of the set concerned, each element exactly once. Where it is the empty set, write '*it is the empty set*'.

(a) $f(\emptyset)$
(b) $f(\{0\})$
(c) $f(\{0, 1\})$
(d) $f(\{0, 1, 2\})$
(e) $f(\{0, 2, 4\})$
(f) $f(A)$
(g) $f^{-1}(\emptyset)$
(h) $f^{-1}(\{5\})$
(i) $f^{-1}(\{6\})$
(j) $f^{-1}(\{5, 6\})$
(k) $f^{-1}(\{5, 6, 7\})$
(l) $f^{-1}(B)$

38. Define the function $f : [-1, 2] \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if} \quad x = -1 \\ x + 1 & \text{if} \quad -1 < x < 0 \\ 2x & \text{if} \quad 0 \leq x \leq 1 \\ -x + 2 & \text{if} \quad 1 < x \leq 2 \end{cases}$$

Write down the respective values of the numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \mu, \nu, \sigma, \tau$, so that the set equalities in parts (a), (b), (c), (d) hold. You are not required to justify your answer. (But it may help if you draw the graph of $f$ first.)

(a) $f^{-1}(\{0\}) = \{\alpha, 2\}$.

(b) $f([1, 2]) = [0, \beta) \cup \{\gamma\}$.

(c) $f^{-1}([0.5, 3]) = \{\delta\} \cup [-0.5, \varepsilon) \cup [\zeta, 1.5]$.

(d) $f^{-1}(f([0, 0.5])) = [\mu, \nu] \cup (\sigma, \tau]$.

39. Define the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x+3 & \text{if} \quad x < -1 \\ -3 & \text{if} \quad x = -1 \\ -2x-2 & \text{if} \quad -1 < x < 0 \\ 3 & \text{if} \quad x = 0 \\ \dfrac{5}{5x+1} & \text{if} \quad x > 0 \end{cases}.$$

Write down the respective values of the numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta$, so that the set equalities below hold. You are not required to justify your answer. (It may help if you draw the graph of $f$ first.)

(a) $f((-3,0)) = ((\alpha,\beta)\backslash\{\gamma\}) \cup \{\delta\}$.

(b) $f^{-1}([-1,4]) = ([-4,\varepsilon]\backslash\{\zeta\}) \cup \{\eta\} \cup [\theta,+\infty)$.

40.$^\diamond$ Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $g(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$.

Write down the respective values of the numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \kappa, \lambda, \mu, \nu, \xi, \rho, \sigma, \tau, \varphi, \psi, \omega$, so that the set equalities below hold. You are not required to justify your answer.

(a) $g(\{1\}) = \{\alpha\}$.

(b) $g(\{\sqrt{2}\}) = \{\beta\}$.

(c) $g(\{1, \sqrt{2}, \sqrt{3}\}) = \{\gamma, \delta\}$.

(d) $g^{-1}(\{1\}) = \{\varepsilon\}$.

(e) $g^{-1}(\{1, \sqrt{2}\}) = \{\zeta\}$.

(f) $g^{-1}(\{0,1\}) = \{\eta, \theta\} \cup (\mathbb{R}\backslash\mathbb{Q})$.

(g) $g([1,2]) = \{\kappa\} \cup ([\lambda,\mu] \cap \mathbb{Q})$.

(h) $g^{-1}(g([1,2])) = \{\nu\} \cup [\xi,\rho] \cup (\mathbb{R}\backslash\mathbb{Q})$.

(i) $g^{-1}([1,3]) = [\sigma,\tau] \cap \mathbb{Q}$.

(j) $g(g^{-1}([1,3])) = [\varphi,\psi] \cap \mathbb{Q}$.

41. *You are not required to justify your answers in this question.*

Let $c \in \mathbb{R}$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} |(x+5)(x-3)| + 2 & \text{if} \quad x \leq 0 \\ \dfrac{c}{1+x^2} & \text{if} \quad x > 0 \end{cases}$$

Suppose $f^{-1}(\{v\}) \neq \emptyset$ for any $v \in (0,+\infty)$. Further suppose $f^{-1}(\{2\})$ has exactly one element.

(a) What is the value of $c$?

(b) Name the only element of $f^{-1}(\{2\})$.

(c) What are the numbers $\alpha, \beta, \gamma, \delta$ for which the set equality $f([-3,3]) = [\alpha,\beta) \cup [\gamma,\delta]$ holds?

42. *You are not required to justify your answers in this question.*

Let $a, b \in \mathbb{R}$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \dfrac{2}{1+x^2} + b & \text{if} \quad x \leq 0 \\ x & \text{if} \quad 0 < x < 2 \\ a & \text{if} \quad x = 2 \\ \dfrac{1}{x-2} & \text{if} \quad x > 2 \end{cases}$$

Suppose $f(\mathbb{R}) = [0,+\infty)$. Further suppose that the set $f^{-1}(\{1\})$ has exact two elements, and for any $v \in (3,+\infty)$, the set $f^{-1}(\{v\})$ is a singleton.

(a) What is the value of $a$?

(b) Write down both elements of $f^{-1}(\{1\})$.

(c) What is the value of $b$?

(d) What are the numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ for which the set equality $f^{-1}((1.5,2]) = (\alpha,\beta] \cup (\gamma,\delta) \cup [\varepsilon,\zeta)$ holds?

43. Let $f : \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{1}{x^2}$ for any $x \in \mathbb{R}\backslash\{0\}$.

(a) Consider the sets below. Express each of them as an interval or a union of several 'disjoint' intervals.

i. $f([1,2])$ ii. $f((0,1))$ iii. $f^{-1}([1,4])$ iv. $f^{-1}([-1,1])$

(b)$^\diamondsuit$ Prove the set equalities you have written down.

44. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{8x}{x^2+4}$ for any $x \in \mathbb{R}$. Note that $f$ is differentiable on $\mathbb{R}$.

(a) i. Find the respective value of $\lim\limits_{t\to-\infty} f(t)$, $\lim\limits_{t\to+\infty} f(t)$.

ii. Determine where $f$ is strictly increasing, and where $f$ is strictly decreasing.

iii. Hence, or otherwise, determine where $f$ attains any relative maximum or any relative minimum, if such exist.

iv. Hence, or otherwise, determine where $f$ attains the absolute maximum or the absolute minimum, if such exists.

**Remark.** With these information, you should be able to sketch the graph of $f$.

(b) i. Consider the sets below. Express each of them as an interval or a union of several 'disjoint' intervals.

A. $f(\mathbb{R})$ B. $f([0,1])$ C. $f([1,3])$ D. $f([1,+\infty))$

ii.$^\clubsuit$ Prove the set equalities you have written down.

**Remark.** The difficulty is in the juggling of inequalities (involving fractions and surd forms) together with quantifiers.

(c) i. Consider the sets below. Express each of them as an interval or a union of several 'disjoint' intervals.

A. $f^{-1}(\mathbb{R})$ B. $f^{-1}([0,2])$ C. $f^{-1}((1,3))$ D. $f^{-1}((-1,1))$

ii.$^\diamondsuit$ Prove the set equalities you have written down.

45. (a)$^\diamondsuit$ Let $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ be defined by $f(t) = \left(\dfrac{2t}{t^2+1}, \dfrac{t^2-1}{t^2+1}\right)$ for any $t \in \mathbb{R}$. Let $C = \{(u,v) \mid u,v \in \mathbb{R} \text{ and } u^2+v^2 = 1\}$ and $p = (0,1)$.

i. Verify that $f$ is injective. ii. Verify that $f(\mathbb{R}) = C \backslash \{p\}$.

**Remark.** Hence the function $f$ parametrizes the circle $C$ with the point $p$ removed. It is the 'baby version' of what is known as the **stereographic projection** for the ($n$-dimensional) sphere. The stereographic projection for the (2-dimensional) sphere is described in the next part, with $\mathbb{R}^2$ being identified as $\mathbb{C}$.

(b)$^\diamondsuit$ Let $g : \mathbb{C} \longrightarrow \mathbb{R}^3$ be defined by $g(z) = \left(\dfrac{2\mathsf{Re}(z)}{|z|^2+1}, \dfrac{2\mathsf{Im}(z)}{|z|^2+1}, \dfrac{|z|^2-1}{|z|^2+1}\right)$ for any $z \in \mathbb{C}$. Let $S = \{(u,v,w) \mid u,v,w \in \mathbb{R} \text{ and } u^2+v^2+w^2 = 1\}$ and $p = (0,0,1)$.

i. Verify that $g$ is injective. ii. Verify that $g(\mathbb{C}) = S \backslash \{p\}$.

46.$^\diamondsuit$ Let $r > 0$ and $R > 0$. Suppose $R > r$. Let $f : (-1,1) \times (-1,1) \longrightarrow \mathbb{R}^3$ be defined by $f(s,t) = ((R + r\cos(\pi t))\cos(\pi s), (R + r\cos(\pi t))\sin(\pi s), r\sin(\pi t))$ for any $s,t \in (-1,1)$.

Let $T = \{(x,y,z) \mid x,y,z \in \mathbb{R} \text{ and } (\sqrt{x^2+y^2} - R)^2 + z^2 = r^2\}$, $C_1 = \{(x,0,z) \mid x,z \in \mathbb{R} \text{ and } (x+R)^2 + z^2 = r^2\}$, $C_2 = \{(x,y,0) \mid x,y \in \mathbb{R} \text{ and } x^2+y^2 = (R-r)^2\}$.

(a) Verify that $f$ is injective. (b) Verify that $f((-1,1) \times (-1,1)) = T \backslash (C_1 \cup C_2)$.

**Remark.** $T$ is a 'torus' in $\mathbb{R}^3$ with centre at the origin, with the $z$-axis as the axis of symmetry, and the $xy$-plane as the plane of symmetry. It is the surface of revolution obtained by rotating the circle $(x-R)^2 + z^2 = r^2$ on the $xz$-plane around the $z$-axis.

47.$^\diamondsuit$ Let $f, g, h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be functions defined by $f(x,y) = |x| + |y|$, $g(x,y) = \sqrt{x^2+y^2}$, $h(x,y) = \max(|x|, |y|)$ for any $x, y \in \mathbb{R}$.

Let $I = [0,1]$. Prove that $f^{-1}(I) \subsetneq g^{-1}(I) \subsetneq h^{-1}(I)$.

**Remark.** It helps to sketch the 'contour map' on $\mathbb{R}^2$ for the functions $f, g, h$ first.

48.$^\clubsuit$ Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = z^2$ for any $z \in \mathbb{C}$. Let $L = \{z \in \mathbb{C} : \mathsf{Re}(z) = 1\}$.

(a) Let $\gamma : \mathbb{R} \longrightarrow \mathbb{C}$ be the function defined by $\gamma(t) = 1 + ti$ for any $t \in \mathbb{R}$.

i. Prove that $\gamma(\mathbb{R}) = L$.

ii. Find the 'explicit formula of definition' for the function $f \circ \gamma$.

iii. Hence, or otherwise, prove that $f(L)$ is the set of all points of a certain parabola on the Argand plane. Also give the equation for that parabola.

(b) i. Let $t \in \mathbb{R}$. Suppose $z = \zeta$ is a solution of the equation $z^2 = 1 + ti$ with unknonwn $z \in \mathbb{C}$. Prove that $(\mathsf{Re}(\zeta))^2 - (\mathsf{Im}(\zeta))^2 = 1$ and $2\mathsf{Re}(\zeta)\mathsf{Im}(\zeta) = t$.

ii. Prove that $f^{-1}(L)$ is the set of all points of a certain hyperbola (with two branches) on the Argand plane. Also give the equation for that hyperbola.

**Remark.** How about replacing $L$ by some other straight line on the Argand plane? (Be careful with the difference between the case in which the straight line passes through 0 and the case in which the straight line does not pass through 0.)

49.♣ Let $a \in \mathbb{R}\backslash\{0\}$, $A = \{w \in \mathbb{C} : \mathsf{Im}(w) = a\}$, and $f : \mathbb{C}\backslash\{0\} \longrightarrow \mathbb{C}\backslash\{0\}$ be the function defined by $f(z) = \dfrac{1}{z}$ for any $z \in \mathbb{C}\backslash\{0\}$.

(a) Prove that $f^{-1}(A) = \left\{z \in \mathbb{C}\backslash\{0\} : (\mathsf{Re}(z))^2 + (\mathsf{Im}(z))^2 + \dfrac{\mathsf{Im}(z)}{a} = 0\right\}$.

(b) Sketch the picture of $f^{-1}(A)$ as a ‘geometric figure’ on the ‘Argand plane’ of $\mathbb{C}$.

50.♢ Denote by $C^\infty(\mathbb{R})$ the set of all smooth real-valued functions of one real variable whose respective domains are $\mathbb{R}$.
For each $n \in \mathbb{N}$, denote by $P_n$ the set of all degree-$n$ polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ with real coefficients.
Define the function $D : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ by $D(\varphi) = \varphi'$ for any $\varphi \in C^\infty(\mathbb{R})$.
Verify that $D(P_{n+1}) = P_n$ for any $n \in \mathbb{N}$.

51.♢ *Familiarity with linear algebra is assumed in this question.*
We introduce the following definition:

- *Let $S$ be a subset of $\mathbb{R}^n$. The set $S$ is said to be* **convex** *if for any $p, q \in \mathbb{R}^n$, for any $t \in [0, 1]$, $tp + (1 - t)q \in S$.*

Let $H$ be an $m \times n$-matrix with real entries, and $L_H : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the function defined by $L_H(x) = Hx$ for any $x \in \mathbb{R}^n$. (Note that $L_H$ is a linear transformation: we have $L_H(ax + by) = aL_H(x) + bL_H(y)$ for any $x, y \in \mathbb{R}^n$.)

(a) Prove that $L_H(S)$ is a convex subset of $\mathbb{R}^m$ for any convex subset $S$ of $\mathbb{R}^n$.

(b) Prove that ${L_H}^{-1}(U)$ is a convex subset of $\mathbb{R}^n$ for any convex subset $U$ of $\mathbb{R}^m$.

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52. (a) Let $F = \{(x, y) \mid y^3 = x^2 + 1 \text{ and } x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$. (Note that $F \subset \mathbb{R}^2$.)
Verify that $(\mathbb{R}, \mathbb{R}, F)$ is a function.

(b) Let $A = (0, +\infty)$, and $F = \{(x, y) \mid x^2y = 1 \text{ and } x \in A \text{ and } y \in A\}$. (Note that $F \subset A^2$.)
Verify that $(A, A, F)$ is a function.

(c) Let $A_1 = [0, +\infty)$, $A_2 = (-\infty, 0)$, $F_1 = \{(x, y) \mid y = x^2 \text{ and } x \in A_1 \text{ and } y \in \mathbb{R}\}$, $F_2 = \{(x, y) \mid y = x+1 \text{ and } x \in A_2 \text{ and } y \in \mathbb{R}\}$, and $F = F_1 \cup F_2$. (You can take for granted that $F \subset \mathbb{R}^2$.)
Verify that $(\mathbb{R}, \mathbb{R}, F)$ is a function.
**Remark.** Such is an example of ‘piecewise-defined functions’ in school mathematics.

53. (a) Let $G = \{(x, y) \mid |y| = |x| \text{ and } x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$. Is $(\mathbb{R}, \mathbb{R}, G)$ a function? Justify your answer.

(b) Let $H = \{(x, y) \mid y^2 = x \text{ and } x \in \mathbb{N} \text{ and } y \in \mathbb{N}\}$. Is $(\mathbb{N}, \mathbb{N}, H)$ a function? Justify your answer.

(c) Let $G = \{(\sqrt{t^2}, \cos(t)) \mid t \in \mathbb{R}\}$. Is $(\mathbb{R}, \mathbb{R}, G)$ a function? Justify your answer.

(d) Let $H = \{(s^2, s) \mid s \in \mathbb{R}\} \cup \{(t, 0) \mid t \in \mathbb{R} \text{ and } t < 0\}$. Is $(\mathbb{R}, \mathbb{R}, H)$ a function? Justify your answer.

54. In this question, $0, 1, 2$ are regarded as pairwise distinct objects.

(a) Let $A = \{0, 1, 2\}$, $B = \{0, 1, 2\}$, and $F = \{(0, 1), (1, 2), (2, 1)\}$. Note that $F \subset A \times B$.
Is $(A, B, F)$ a function? Justify your answer.

(b) Let $A = \{0, 1\}$, $B = \{0, 1, 2\}$, and $F = \{(0, 1), (0, 2), (1, 0)\}$. Note that $F \subset A \times B$.
Is $(A, B, F)$ a function? Justify your answer.

(c) Let $A = \{0, 1, 2\}$, $B = \{0, 1, 2\}$, and $F = \{(0, 0), (1, 2)\}$. Note that $F \subset A \times B$.
Is $(A, B, F)$ a function? Justify your answer.

55. (a) Let $A = [0, 1]$, $B = [0, 2]$, and $F = \{(x, y) \mid x \in A \text{ and } y \in B \text{ and } 4x^2 + y^2 = 4\}$. Note that $F \subset A \times B$.
Define $f = (A, B, F)$. Verify that $f$ is a function.

(b) Let $A = [0,1]$, $B = [0,1]$, and $F = \{(x,y) \mid x \in A \text{ and } y \in B \text{ and } 4x^2 + y^2 = 4\}$. Note that $F \subset A \times B$. Define $f = (A,B,F)$. Is $f$ a function? Justify your answer.

(c) Let $A = [0,1]$, $B = [-1,2]$, and $F = \{(x,y) \mid x \in A \text{ and } y \in B \text{ and } 4x^2 + y^2 = 4\}$. Note that $F \subset A \times B$. Define $f = (A,B,F)$. Is $f$ a function? Justify your answer.

56. (a) Let $A = [0,3]$, $B = [0,2]$, and $F = \{(x,y) \mid x \in A \text{ and } y \in B \text{ and } 4x^2 + 9y^2 = 36\}$. Note that $F \subset A \times B$. Define $f = (A,B,F)$. Verify that $f$ is a function.

(b) Let $A = [0,3]$, $B = [-1,2]$, and $F = \{(x,y) \mid x \in A \text{ and } y \in B \text{ and } 4x^2 + 9y^2 = 36\}$. Note that $F \subset A \times B$. Define $f = (A,B,F)$. Is $f$ a function? Justify your answer.

(c) Let $A = [0,3]$, $B = [0,1]$, and $F = \{(x,y) \mid x \in A \text{ and } y \in B \text{ and } 4x^2 + 9y^2 = 36\}$. Note that $F \subset A \times B$. Define $f = (A,B,F)$. Is $f$ a function? Justify your answer.

57. (a) Let $A = [0,+\infty)$, $F = \{(x,y) \mid x \in A \text{ and } y \in \mathbb{R} \text{ and } (y+2)^2 = x\}$. Note that $F \subset A \times \mathbb{R}$. Is $(A, \mathbb{R}, F)$ a function? Justify your answer.

(b) Let $A = [0,+\infty)$, $F = \{(x,y) \mid x \in A \text{ and } y \in A \text{ and } (y+2)^2 = x\}$. Note that $F \subset A^2$. Is $(A,A,F)$ a function? Justify your answer.

(c) Let $A = [0,+\infty)$, $B = [-2,+\infty)$, $F = \{(x,y) \mid x \in A \text{ and } y \in B \text{ and } (y+2)^2 = x\}$. Note that $F \subset A \times B$. Is $(A,B,F)$ a function? Justify your answer.

58. Let $A = [0,2]$, $B = [0,3]$, and $F = \{(x,y) \mid x \in A \text{ and } y \in B \text{ and } y^2 = x^2(2-x)\}$. Note that $F \subset A \times B$ by definition. Define $f = (A,B,F)$.

(a) Verify that $f$ is a function.

(b) Is $f$ injective? Justify your answer.

59. Let $A = [0,+\infty)$. Let $\alpha, \beta \in \mathbb{R}$. Define $C_{\alpha,\beta} = \{(x,y) \mid (y-\beta)^2 = 1 + (x-\alpha)^3\}$.

(a) Define $E_\alpha = A^2 \cap C_{\alpha,0}$.

i.♣ For which values of $\alpha$ is $(A,A,E_\alpha)$ a function? Justify your answer.

ii.♢ For which values of $\alpha$ is $(A,A,E_\alpha)$ an injective function? Justify your answer.

iii.♣ For which values of $\alpha$ is $(A,A,E_\alpha)$ a surjective function? Justify your answer.

iv.♢ For which values of $\alpha$ is $(A,A,E_\alpha)$ a bijective function? Justify your answer.

(b) Write $F_\beta = A^2 \cap C_{0,\beta}$.

i.♡ For which values of $\beta$ is $(A,A,F_\beta)$ a function? Justify your answer.

ii.♣ For which values of $\beta$ is $(A,A,F_\beta)$ an injective function? Justify your answer.

iii.♣ For which values of $\beta$ is $(A,A,F_\beta)$ a surjective function? Justify your answer.

iv.♢ For which values of $\beta$ is $(A,A,F_\beta)$ a bijective function? Justify your answer.

60. Let $A = \{x \in \mathbb{Q} : x = s^3 \text{ for some } s \in \mathbb{Q}\}$, $B = \{y \in \mathbb{Q} : y = t^4 \text{ for some } t \in \mathbb{Q}\}$. Define

$$F = \left\{ (x,y) \;\middle|\; \begin{array}{l} x \in A \text{ and } y \in B \text{ and} \\ \text{there exists some } r \in \mathbb{Q} \text{ such that } (x = r^3 \text{ and } y = r^4). \end{array} \right\},$$

and $f = (A,B,F)$. Note that $F \subset A \times B$.

(a)♣ Is $f$ a function from $A$ to $B$? Justify your answer.

(b) Where $f$ is a function, write down the 'formula of definition' of $f$.

(c)♢ Where $f$ is a function, determine whether $f$ is injective. Justify your answer.

(d)♢ Where $f$ is a function, determine whether $f$ is surjective. Justify your answer.

61.♢ For each $n \in \mathbb{N}\backslash\{0\}$, we define $\omega_n = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)$, and $Z_n = \{\zeta \in \mathbb{C} : \zeta = {\omega_n}^k \text{ for some } k \in \mathbb{Z}\}$.

*You may take for granted that* ${\omega_n}^n = 1$ *and* $Z_n = \{1, \omega_n, {\omega_n}^2, \cdots, {\omega_n}^{n-1}\}$. (Note that $Z_n$ is the set of all $n$-th roots of unity.)

Consider the 'declarations' below through each of which some function is supposed to be defined. Determine whether it makes sense or not. Justify your answer.

(a) '*Define the function* $f : Z_6 \longrightarrow \mathbb{Z}$ *by* $f({\omega_6}^k) = 6k$ *for any* $k \in \mathbb{Z}$.'

(b) '*Define the function* $f : Z_{12} \longrightarrow Z_6$ *by* $f({\omega_{12}}^k) = {\omega_6}^k$ *for any* $k \in \mathbb{Z}$.'

62. For any $k \in \mathbb{N}\backslash\{0\}$, define $\omega_k = \cos\left(\frac{2\pi}{k}\right) + i\sin\left(\frac{2\pi}{k}\right)$, and define $Z_k = \{\zeta \in \mathbb{C} : \zeta = {\omega_k}^j \text{ for some } j \in \mathbb{Z}\}$.

Let $m, n \in \mathbb{N}\backslash\{0\}$. Define the subset $F_{m,n}$ of $Z_m \times Z_n$ by

$$F_{m,n} = \{(\zeta, \eta) \mid \text{ There exist some } r \in \mathbb{Z} \text{ such that } \zeta = {\omega_m}^r \text{ and } \eta = {\omega_n}^r \ \}.$$

Define the relation $f_{m,n}$ by $f_{m,n} = (Z_m, Z_n, F_{m,n})$. Consider the statements $(\star), (\star\star)$ below:

$(\star)$ *$m$ is divisible by $n$.*

$(\star\star)$ *$f_{m,n}$ is a function from $Z_m$ to $Z_n$.*

(a)$^\heartsuit$ Suppose $(\star)$ holds. Prove that $(\star\star)$ holds.

(b)$^\clubsuit$ Suppose $(\star\star)$ holds. Prove that $(\star)$ holds.

63. Let $A = \mathbb{C}\backslash\{1\}$, $F = \{(x, y) \mid x \in A \text{ and } y \in A \text{ and } (x-1)y = x\}$. Define $f = (A, A, F)$. Note that $F \subset A^2$.

(a) Verify that $f$ is a function.

(b) What is the 'formula of definition' of the function $f$.

(c) Verify that the function $f$ is bijective.

(d) What is the 'formula of definition' of the inverse function of $f$.

(e) What are the respective 'formulae of definition' of $f \circ f$ and $f \circ f \circ f$?

64.$^\diamondsuit$ Prove the statement below:

- *Let $A, B, C$ be sets and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. For any subset $S$ of $A$, $(g \circ f)(S) = g(f(S))$.*

65. (a)$^\diamondsuit$ Prove the statement below:

  - *Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. For any subsets $U, V$ of $B$, if $U \subset V$ then $f^{-1}(U) \subset f^{-1}(V)$.*

(b)$^\clubsuit$ Dis-prove the statement below:

  - *Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. For any subsets $U, V$ of $B$, if $f^{-1}(U) \subset f^{-1}(V)$ then $U \subset V$.*

(c)$^\clubsuit$ *You may apply the result obtained in the first part of this question to help simplify your arguments in this part.*
Prove the statements below:

  i. *Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. For any subsets $U, V$ of $B$, $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$.*

  ii. *Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. For any subsets $U, V$ of $B$, $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$.*

66. (a)$^\diamondsuit$ Prove the statement below:

  - *Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. For any subsets $S, T$ of $A$, $f(S \cap T) \subset f(S) \cap f(T)$.*

(b)$^\diamondsuit$ Dis-prove the statement below:

  - *Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. For any subsets $S, T$ of $A$, $f(S) \cap f(T) \subset f(S \cap T)$.*

(c)$^\clubsuit$ Prove the statements below:

  i. *Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. Suppose that for any subsets $S, T$ of $A$, $f(S) \cap f(T) \subset f(S \cap T)$. Then $f$ is injective.*

  ii. *Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. Suppose $f$ is injective. Then for any subsets $S, T$ of $A$, $f(S) \cap f(T) \subset f(S \cap T)$.*

67. (a)$^\diamondsuit$ Prove the statement below:

  - *Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. For any subset $S$ of $A$, $S \subset f^{-1}(f(S))$.*

(b)$^\diamondsuit$ Dis-prove the statement below:

  - *Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. For any subset $S$ of $A$, $f^{-1}(f(S)) \subset S$.*

(c)$^\clubsuit$ Prove the statements below:

  i. *Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. Suppose that for any subset $S$ of $A$, $f^{-1}(f(S)) \subset S$. Then $f$ is injective.*

  ii. *Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. Suppose $f$ is injective. Then for any subset $S$ of $A$, $f^{-1}(f(S)) \subset S$.*

68.$^\clubsuit$ *You may apply results already established in the lectures or in other exercises to help simplify the argument here.*
Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. Prove the following statements:

(a) $f(S) = f(f^{-1}(f(S)))$ *for any subset* $S$ *of* $A$.

(b) $f^{-1}(U) = f^{-1}(f(f^{-1}(U)))$ *for any subset* $U$ *of* $B$.

69.$^\clubsuit$ Prove the statement below:

- *Let* $A, B$ *be sets, and* $f : A \longrightarrow B$ *be a function. For any subset* $S$ *of* $A$*, for any subset* $U$ *of* $B$*,* $f(S \cap f^{-1}(U)) = f(S) \cap U$.

70.$^\spadesuit$ We introduce the definitions and notations below:

- *Let* $A, B$ *be sets and* $f : A \longrightarrow B$ *be a function.*
  $f$ *induces the pair of functions* $f_{\mathfrak{P}} : \mathfrak{P}(A) \longrightarrow \mathfrak{P}(B)$, $f^{\mathfrak{P}} : \mathfrak{P}(B) \longrightarrow \mathfrak{P}(A)$*, given by* $f_{\mathfrak{P}}(S) = f(S)$ *for any* $S \in \mathfrak{P}(A)$, $f^{\mathfrak{P}}(U) = f^{-1}(U)$ *for any* $U \in \mathfrak{P}(B)$ *respectively.*

(a) Let $f : A \longrightarrow B$, $g : A \longrightarrow B$ be functions. Prove that the statements (A), (B), (C) are logically equivalent:

(A) $f = g$.

(B) $f_{\mathfrak{P}} = g_{\mathfrak{P}}$.

(C) $f^{\mathfrak{P}} = g^{\mathfrak{P}}$.

(b) Let $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Prove the statements below:

i. $(g \circ f)_{\mathfrak{P}} = g_{\mathfrak{P}} \circ f_{\mathfrak{P}}$.

ii. $(g \circ f)^{\mathfrak{P}} = f^{\mathfrak{P}} \circ g^{\mathfrak{P}}$.

(c) Let $f : A \longrightarrow B$ be a function. Prove that the statements (D), (E), (F) are logically equivalent:

(D) $f$ *is injective.*

(E) $f_{\mathfrak{P}}$ *is injective.*

(F) $f^{\mathfrak{P}}$ *is surjective.*

71.$^\diamond$ Let $A, B$ be sets, and $f, g : A \longrightarrow B$ be functions, with graphs $F, G$ respectively. Prove that the statements (†), (‡) are equivalent to each other:

(†) $f(x) = g(x)$ *for any* $x \in A$.

(‡) $F = G$.

**Remark.** What is the point of this question? We have introduced two notions of 'equality of functions': equality of functions as ordered triples, and equality of functions as 'assignments from $A$ to $B$'. Are they in conflict? By virtue of the equivalence of the statements (†), (‡), these two notions of 'equality of functions' are consistent.

72.$^\diamond$ Prove the statements below:

(a) *Let* $A, B$ *be sets, and* $f : A \longrightarrow B$, $g : B \longrightarrow A$, $h : B \longrightarrow A$ *be functions. Supppose* $g \circ f = \mathsf{id}_A$ *and* $f \circ h = \mathsf{id}_B$*. Then* $f$ *is bijective, and* $f^{-1} = g = h$.

(b) *Let* $A, B$ *be sets, and* $f : A \longrightarrow B$ *be a bijective function.* $(f^{-1})^{-1} = f$.

(c) *Let* $A, B, C$ *be sets, and* $f : A \longrightarrow B$, $g : B \longrightarrow C$ *be bijective functions.* $g \circ f$ *is a bijective function and* $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

73. (a) Prove the statements below:

i. *Let* $A, B$ *be non-empty sets, and* $f : A \longrightarrow B$ *be a function. Suppose there exists some function* $g : B \longrightarrow A$ *such that* $g \circ f = \mathsf{id}_A$*. Then* $f$ *is injective.*

ii.$^\diamond$ *Let* $A, B$ *be non-empty sets, and* $f : A \longrightarrow B$ *be a function. Suppose* $f$ *is injective. Then there exists some function* $g : B \longrightarrow A$ *such that* $g \circ f = \mathsf{id}_A$.

**Remark.** To write down a simple argument, we need the notion of bijective function.

(b) Prove the statements below:

i. *Let* $A, B$ *be non-empty sets, and* $f : A \longrightarrow B$ *be a function. Suppose there exists some function* $g : B \longrightarrow A$ *such that* $f \circ g = \mathsf{id}_B$*. Then* $f$ *is surjective.*

ii.$^\diamond$ *Let* $A, B$ *be non-empty sets, and* $f : A \longrightarrow B$ *be a function. Suppose* $f$ *is surjective. Then there exists some function* $g : B \longrightarrow A$ *such that* $f \circ g = \mathsf{id}_B$.

**Remark.** Re-read your argument after finishing it. Do you believe you have indeed proved anything?

(c) Dis-prove the statements below:

i. *Let* $A, B$ *be non-empty sets, and* $f : A \longrightarrow B$ *be a function. Suppose there exists some function* $g : B \longrightarrow A$ *such that* $g \circ f = \mathsf{id}_A$*. Then* $f$ *is surjective.*

ii. *Let* $A, B$ *be non-empty sets, and* $f : A \longrightarrow B$ *be a function. Suppose there exists some function* $g : B \longrightarrow A$ *such that* $f \circ g = \mathsf{id}_B$*. Then* $f$ *is injective.*

74. Consider each of the statements below. For each of them, determine whether it is true or false. Justify your answer by giving an appropriate argument.

(a) *Let $A, B$ be non-empty sets, and $f : A \longrightarrow B$ be a function. Suppose $f$ is injective. Then there exists some function $g : B \longrightarrow A$ such that $f \circ g = \mathsf{id}_B$.*

(b) *Let $A, B$ be non-empty sets, and $f : A \longrightarrow B$ be a function. Suppose $f$ is surjective. Then there exists some function $g : B \longrightarrow A$ such that $g \circ f = \mathsf{id}_A$.*

(c)$^\clubsuit$ *Let $A$ be a non-empty set, and $f : A \longrightarrow A$ be a function. Suppose there exists some function $g : A \longrightarrow A$ such that $g \circ f = \mathsf{id}_A$. Then $f$ is surjective.*

(d)$^\clubsuit$ *Let $A$ be a non-empty set, and $f : A \longrightarrow A$ be a function. Suppose there exists some function $g : A \longrightarrow A$ such that $f \circ g = \mathsf{id}_A$. Then $f$ is injective.*

(e)$^\diamondsuit$ *Let $A$ be a non-empty set, and $f : A \longrightarrow A$ be a function. Suppose $f$ is injective. Then there exists some function $g : A \longrightarrow A$ such that $f \circ g = \mathsf{id}_A$.*

(f)$^\diamondsuit$ *Let $A$ be non-empty set, and $f : A \longrightarrow A$ be a function. Suppose $f$ is surjective. Then there exists some function $g : A \longrightarrow A$ such that $g \circ f = \mathsf{id}_A$.*

75.$^\diamondsuit$ Let $A, B, F$ be sets. Suppose $f = (A, B, F)$ is a function. Define $\hat{f} = (A, f(A), F)$. Prove the statements below:

(a) $F \subset A \times f(A)$.

(b) *$\hat{f}$ is a surjective function.*

(c) *Suppose $f$ is injective, and suppose $H = \{(f(y), y) \mid y \in A\}$. Then $\hat{f}$ is a bijective function, with the inverse function of $\hat{f}$ being given by $(f(A), A, H)$.*

76.$^\spadesuit$ We introduce/recall some definitions:

- *Let $A, B$ be sets, and $g : A \longrightarrow B$ be a function. Suppose $C$ is a subset of $A$. The function $g\big|_C : C \longrightarrow B$ defined by $g\big|_C(x) = g(x)$ for any $x \in A$ is called the* **restriction of $g$ to $C$**.
- *Let $A, B$ be sets. The function $\pi_A : A \times B \longrightarrow A$ defined by $\pi_A(x, y) = x$ for any $x \in A$, $y \in B$ is called the* **projection function from $A \times B$ to $A$**. *The function $\pi_B : A \times B \longrightarrow B$ defined by $\pi_B(x, y) = y$ for any $x \in A$, $y \in B$ is called the* **projection function from $A \times B$ to $B$**.

Let $f = (A, B, G)$ be a relation. Consider the respective projection functions $\pi_A : A \times B \longrightarrow A$, $\pi_B : A \times B \longrightarrow B$, and the respective restrictions $\pi_A\big|_G : G \longrightarrow A$, $\pi_B\big|_G : G \longrightarrow B$.

(a) Prove the statement below:

- The relation $f$ is a function iff the function $\pi_A\big|_G$ is bijective.

(b) Now suppose the relation $f$ is a function. Prove the statements below:

i. $\pi_B\big|_G = f \circ \pi_A\big|_G$.

ii. *The function $f$ is surjective iff the function $\pi_B\big|_G$ is surjective.*

iii. *The function $f$ is injective iff the function $\pi_B\big|_G$ is injective.*

iv. *The function $f$ is bijective iff the function $\pi_B\big|_G$ is bijective.*

77.$^\spadesuit$ We introduce the notation below:

- *Let $D, R$ be sets. The* **set of all functions from $D$ to $R$** *is denoted by $\mathsf{Map}(D, R)$.*

(a) Prove the statements below:

i. *Let $B$ be a set. The set $\mathsf{Map}(\emptyset, B)$ is a singleton.*

ii. *Let $A, B$ be sets. Suppose $A, B$ are non-empty. Then $\mathsf{Map}(A, B)$ is non-empty.*

iii. *Let $A, B$ be sets. Suppose $A$ is non-empty. Then $\mathsf{Map}(A, \emptyset) = \emptyset$.*

(b) Let $A, B$ be non-empty sets, and $f : A \longrightarrow B$ be a function. Prove that the statements $(\dagger_1), (\ddagger_1)$ are equivalent:

$(\dagger_1)$ *$f$ is surjective.*

$(\ddagger_1)$ *For any non-empty set $C$, for any $\varphi, \psi \in \mathsf{Map}(B, C)$, (if $\varphi \circ f = \psi \circ f$ then $\varphi = \psi$).*

(c) Let $A, B$ be non-empty sets, and $f : A \longrightarrow B$ be a function. Prove that the statements $(\dagger_2), (\ddagger_2)$ are equivalent:

$(\dagger_2)$ *$f$ is injective.*

$(\ddagger_2)$ *For any non-empty set $C$, for any $\varphi, \psi \in \mathsf{Map}(C, A)$, (if $f \circ \varphi = f \circ \psi$ then $\varphi = \psi$).*

(d) Let $A, B$ be non-empty sets, and $f : A \longrightarrow B$ be a function. Prove that the statements $(\dagger_3), (\ddagger_3)$ are equivalent:

$(\dagger_3)$ *$f$ is surjective.*

$(\ddagger_3)$ *For any non-empty set $C$, for any $\psi \in \mathsf{Map}(C, B)$, there exists some $\varphi \in \mathsf{Map}(C, A)$ such that $f \circ \varphi = \psi$.*

(e) Let $A, B$ be non-empty sets, and $f : A \longrightarrow B$ be a function. Prove that the statements $(\dagger_4), (\ddagger_4)$ are equivalent:

$(\dagger_4)$ *$f$ is injective.*

$(\ddagger_4)$ *For any non-empty set $C$, for any $\varphi \in \mathsf{Map}(A, C)$, there exists some $\psi \in \mathsf{Map}(B, C)$ such that $\psi \circ f = \varphi$.*

78. *In this question, you may take for granted the validity of the* **Bounded-Monotone Theorem** *and the* **Sandwich Rule** *for infinite sequences of real numbers, and other standard results about limits and continuity from, say, MATH1010. We are going to apply these results to prove the* **Intermediate Value Theorem**.

(a) Let $a, b \in \mathbb{R}$, with $a < b$. Let $g : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose $g(a) < 0 < g(b)$.
Further suppose $g(x) \neq 0$ for any $x \in [a, b]$.
Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$ be three recursively defined infinite sequences of real numbers, as constructed below:

- Define $a_0 = a$, $b_0 = b$. Define $c_0 = \dfrac{a_0 + b_0}{2}$.
- For each $n \in \mathbb{N}$, define $a_{n+1}, b_{n+1}, c_{n+1}$ in terms of $a_n, b_n, c_n$ as described below:
  $(*)$ (Case 1). Suppose $g(c_n) < 0$. Then define $a_{n+1} = c_n$, $b_{n+1} = b_n$.
  $(*)$ (Case 2). Suppose $g(c_n) > 0$. Then define $a_{n+1} = a_n$, $b_{n+1} = c_n$.
  Define $c_{n+1} = \dfrac{a_{n+1} + b_{n+1}}{2}$.

i.♣ Apply mathematical induction to prove that for any $n \in \mathbb{N}$,

A. $a_n \leq c_n \leq b_n$, B. $b_n - a_n = \dfrac{b - a}{2^n}$, C. $g(a_n) < 0$ and $g(b_n) > 0$.

ii. Prove that $\{a_n\}_{n=0}^{\infty}$ is increasing

iii. Prove that $\{b_n\}_{n=0}^{\infty}$ is decreasing.

iv. Prove that $\{a_n\}_{n=0}^{\infty}$ is bounded above by $b$

v. Prove that $\{b_n\}_{n=0}^{\infty}$ is bounded below by $a$.

vi.♣ Deduce that the limits $\lim\limits_{n\to\infty} a_n$, $\lim\limits_{n\to\infty} b_n$, $\lim\limits_{n\to\infty} c_n$ exist in $\mathbb{R}$. Also prove that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n$.
**Remark.** From now on we denote the common value of these three limits by $\ell$.

vii. Is it true that $g(\ell) \neq 0$? Justify your answer.

viii.♣ Is it true that $g$ is continuous at $\ell$? Justify your answer.

(b) Hence, or otherwise, deduce **Bolzano's Intermediate Value Theorem**:

- *Let $a, b \in \mathbb{R}$, with $a < b$. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose $f(a)f(b) < 0$. Suppose $f$ is continuous on $[a, b]$. Then there exists some $x_0 \in (a, b)$ such that $f(x_0) = 0$.*

(c) Hence, or otherwise, deduce the **Intermediate Value Theorem**:

- *Let $a, b \in \mathbb{R}$, with $a < b$. Let $h : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose $h(a) \neq h(b)$. Suppose $h$ is continuous on $[a, b]$. Then, for any $\gamma \in \mathbb{R}$, if $\gamma$ is strictly between $h(a)$ and $h(b)$ then there exists some $c \in (a, b)$ such that $h(c) = \gamma$.*

79. *You are assumed to be familiar with the basic properties of continuity in this question.*
Let $f : [0, 1] \longrightarrow \mathbb{R}$ be a function. Suppose $f$ is continuous on $[0, 1]$, and $x^2 + (f(x))^2 = 1$ for any $x \in [0, 1]$.

(a) Let $p \in [0, 1]$. Prove that $f(p) = 0$ iff $p = 1$.

(b) Suppose $f(0) \geq 0$.

i. What is the value of $f(0)$?

ii.♣ Prove that $f(x) = \sqrt{1 - x^2}$ for any $x \in [0, 1]$.
**Remark.** You will need Bolzano's Intermediate Value Theorem at some stage in the argument.

(c)◇ Suppose $f(0) \leq 0$. Prove that $f(x) = -\sqrt{1 - x^2}$ for any $x \in [0, 1]$.

80.◇ *You are assumed to be familiar with the basic properties of continuity in this question.*
Let $f : [0, 1] \longrightarrow \mathbb{R}$ be a function. Suppose $f$ is continuous on $[0, 1]$ Suppose $f(x)$ is a rational number for any $x \in [0, 1]$. By applying the Intermediate Value Theorem, or otherwise, prove that $f$ is constant on $[0, 1]$.

81.◇ *You are assumed to be familiar with the basic properties of continuity in this question.*

(a) Applying Bolzano's Intermediate Value Theorem, or otherwise, prove the statement $(\sharp)$:

$(\sharp)$ *Let $f, g : [0, 1] \longrightarrow \mathbb{R}$ be functions. Suppose $f, g$ are continuous on $[0, 1]$. Suppose $f(0) < g(0)$ and $f(1) > g(1)$. Then there exists some $x_0 \in (0, 1)$ such that $f(x_0) = g(x_0)$.*

(b) Hence deduce the statement ($\flat$):

($\flat$) *Let $f : [0,1] \longrightarrow \mathbb{R}$ be a function. Suppose $f$ is continuous on $[0,1]$. Suppose $0 \leq f(x) \leq 1$ for any $x \in [0,1]$. Then there exists some $x_0 \in [0,1]$ such that $f(x_0) = x_0$.*

**Remark.** This statement is a 'baby version' of **Brouwer's Fixed Point Theorem**. In the context of the conclusion, the number $x_0$ is a **fixed point** of the function $f$, whose domain is $[0,1]$, whose image $f([0,1])$ lies entirely in $[0,1]$, and whose graph lies entirely in the square $[0,1]^2$.

82. *This question is meant to link up what you have learnt in your first course in linear algebra with the notion of functions, and give a preview on what you are going to learn/use in your second course in linear algebra.*

We introduce the definitions and conventions below:

- *We regard $\mathbb{R}^k$ is the set of all $k \times 1$-column vectors with real entries. (Note that $\mathbb{R}^k$ is a vector space over $\mathbb{R}$.) Denote the zero vector in $\mathbb{R}^k$ by $0_k$.*
- *Suppose $A$ is an $m \times n$-matrix with real entries.*
  *We denote the $j$-th column of $A$ by $C_j(A)$ for each $j = 1, 2, \cdots, n$. We regard $C_1(A), C_2(A), \cdots, C_n(A)$ as $m \times 1$-matrix.*
  *We define $L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by $L_A(x) = Ax$ for any $x \in \mathbb{R}^n$. (Recall that $L_A$ is a linear transformation, in the sense that for any $x, w \in \mathbb{R}^n$, for any $\alpha, \beta \in \mathbb{R}$, $L_A(\alpha x + \beta w) = \alpha L_A(x) + \beta L_A(w)$.)*
  *We further define $L_A^\star : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ by $L_A^\star(y) = L_{A^t}(y)$ for any $y \in \mathbb{R}^m$. (Note that $A^t$ is the transpose of the matrix $A$. Also note that by definition, $L_A^\star$ is a linear transformation.)*

Let $H$ be an $m \times n$-matrix with real entries.

(a) i.$^\diamondsuit$ Let $V$ be a vector subspace of $\mathbb{R}^n$. Verify that $L_H(V)$ is a vector subspace of $\mathbb{R}^m$.

ii.$^\diamondsuit$ Let $W$ be a vector subspace of $\mathbb{R}^m$. Verify that ${L_H}^{-1}(W)$ is a vector subspace of $\mathbb{R}^n$.

(b)$^\diamondsuit$ Verify that $\{C_1(H), C_2(H), \cdots, C_n(H)\}$ is a spanning set for the vector space $L_H(\mathbb{R}^n)$.

(c)$^\diamondsuit$ Prove that the statements below are logically equivalent:

(S1) *$L_H$ is surjective.*

(S2) *$\{C_1(H), C_2(H), \cdots, C_n(H)\}$ is a spanning set for the vector space $\mathbb{R}^m$ over $\mathbb{R}$.*

(d)$^\diamondsuit$ Prove that the statements below are logically equivalent:

(I1) *$L_H$ is injective.*

(I2) *For any $x \in \mathbb{R}^n$, if $L_H(x) = 0_m$ then $x = 0_n$.*

(I3) *${L_H}^{-1}(\{0_m\}) = \{0_n\}$.*

(I4) *$\{C_1(H), C_2(H), \cdots, C_n(H)\}$ is linearly independent over $\mathbb{R}$.*

(I5) *$\{C_1(H), C_2(H), \cdots, C_n(H)\}$ is a base for the vector space $L_H(\mathbb{R}^n)$ over $\mathbb{R}$.*

**Remark.** One possible way to proceed is to deduce (I2) from (I1), then (I3) from (I2), then (I1) from (I3), and to further deduce (I4) from (I2), then (I5) from (I4), then (I2) from (I5).

(e) i.$^\diamondsuit$ Verify that $L_H$ is surjective iff $L_H^\star$ is injective.

ii.$^\clubsuit$ Verify that $L_H$ is injective iff $L_H^\star$ is surjective.

**Remark.** You may take for granted the validity of the statement ($*$) which is of fundamental importance in *linear algebra*:

($*$) *Let $A$ be an $m \times n$-matrix with real entries. $\{C_1(A), C_2(A), \cdots, C_n(A)\}$ is a spanning set for $\mathbb{R}^m$ over $\mathbb{R}$ iff $\{C_1(A^t), C_2(A^t), \cdots, C_m(A^t)\}$ is linearly independent over $\mathbb{R}$.*

But how to prove this statement? You need recall what you have learnt in *linear algebra*.[1]

(f) Suppose $L_H$ is bijective.

i.$^\diamondsuit$ Prove that $m = n$.

ii.$^\clubsuit$ Prove that $H$ is an invertible matrix.

**Remark.** You may take for granted that every linearly independent set in $\mathbb{R}^k$ contains at most $k$ vectors as its elements.

[1] The 'direction' '$\Longrightarrow$' is straightforward: apply the statement ($\star$):

($\star$) *For any $x \in \mathbb{R}^k$, if ($w^t x = 0$ for any $w \in \mathbb{R}^k$ then $x = 0_k$.*

(How to prove ($\star$)?)

As for the 'direction' '$\Longleftarrow$', apply mathematical induction to prove the statement ($\dagger$):

($\dagger$) *Let $m \in \mathbb{N}\backslash\{0\}$. For any $n \in \mathbb{N}\backslash\{0\}$, for any $m \times n$ matrix $A$ with real entries, if $\{C_1(A), C_2(A), \cdots, C_n(A)\}$ is a spanning set for $\mathbb{R}^m$ over $\mathbb{R}$ iff $\{C_1(A^t), C_2(A^t), \cdots, C_m(A^t)\}$ is linearly independent over $\mathbb{R}$.*

The mathematical induction is 'done on the number of rows of matrices'. In the argument you will need apply what you have learnt about *systems of linear equations* and *elementary row operations* in *linear algebra*.