

MATH1050 Exercise 6 (Answers and selected solution)

1. Answer.

- (a) The statement $(P \rightarrow R) \rightarrow [(P \rightarrow Q) \wedge (Q \rightarrow R)]$ is neither a tautology nor a contradiction; it is a contingent statement.

P	Q	R	$P \rightarrow Q$	$Q \rightarrow R$	$P \rightarrow R$	$(P \rightarrow Q) \wedge (Q \rightarrow R)$	$(P \rightarrow R) \rightarrow [(P \rightarrow Q) \wedge (Q \rightarrow R)]$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	F
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Remark. Do you see what is wrong in the logic behind the word ‘therefore’ in the argument below?

- If John visits his girlfriend then John puts aside his books. Therefore, if John visits his girlfriend then John plays football; furthermore, if John plays football then John puts aside his books.

- (b) The statement $(P \rightarrow Q) \rightarrow [(P \rightarrow R) \vee (Q \rightarrow R)]$ is neither a tautology nor a contradiction; it is a contingent statement.

P	Q	R	$P \rightarrow Q$	$P \rightarrow R$	$Q \rightarrow R$	$(P \rightarrow R) \vee (Q \rightarrow R)$	$(P \rightarrow Q) \rightarrow [(P \rightarrow R) \vee (Q \rightarrow R)]$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	F	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

2. Answer.

- (a) $\{0, 1\}, \{1, 2, 3\}$.
 (b) $\{0, 1\}, \{1\}, \{1, 2, 3\}, \{3, 4\}, \{\{3\}, \{4\}\}$.
 (c) $\{1\}, \{3, 4\}$.
 (d) $\{1\}, \{3, 4\}, \{\{3\}, \{4\}\}$.
 (e) $\emptyset, \{\{1\}\}, \{\{3, 4\}\}, \{\{1\}, \{3, 4\}\}$.

3. Answer.

- (a) Five. (c) Six. (e) Two. (g) c, r
 (b) Nine. (d) One. (f) One. (h) $\emptyset, \{c\}, \{r\}, \{c, r\}$.

4. Answer.

- (a) $A = \emptyset$.
 (b) $B \neq \emptyset$; 125 is an element of B .

5. (a) Solution.

We proceed to solve the equation (†):

$$\begin{aligned} \cos(2x) &= \sin(x) \quad \text{--- (†)} \\ 1 - 2\sin^2(x) &= \sin(x) \\ 2\sin^2(x) + \sin(x) - 1 &= 0 \\ (\sin(x) + 1)(2\sin(x) - 1) &= 0 \\ \sin(x) = -1 &\quad \text{or} \quad \sin(x) = \frac{1}{2} \end{aligned}$$

(Case 1).

$$\begin{aligned}\sin(x) &= -1 \\ x &= -\frac{\pi}{2} + K \cdot 2\pi \quad \text{where } K \in \mathbb{Z}\end{aligned}$$

(Case 2).

$$\begin{aligned}\sin(x) &= \frac{1}{2} \\ x &= (-1)^M \cdot \frac{\pi}{6} + M\pi \quad \text{where } M \in \mathbb{Z}\end{aligned}$$

The general solution of (†) is given by $x = -\frac{\pi}{2} + K \cdot 2\pi$ where $K \in \mathbb{Z}$, or $x = (-1)^M \cdot \frac{\pi}{6} + M\pi$ where $M \in \mathbb{Z}$.

Define

$$\begin{aligned}A &= \left\{ x \in \mathbb{R} : x = -\frac{\pi}{2} + K \cdot 2\pi \text{ for some } K \in \mathbb{Z} \right\}, \\ B &= \left\{ x \in \mathbb{R} : x = (-1)^M \cdot \frac{\pi}{6} + M\pi \text{ for some } M \in \mathbb{Z} \right\}.\end{aligned}$$

The solution set of (†) is given by $A \cup B$.

(b) **Solution.**

We proceed to solve the equation (†):

$$\begin{aligned}\sin(2x) + \sin(8x) &= \sin(5x) \quad \text{--- (†)} \\ \sin(5x - 3x) + \sin(5x + 3x) &= \sin(5x) \\ 2\sin(5x)\cos(3x) &= \sin(5x) \\ (2\cos(3x) - 1)\sin(5x) &= 0 \\ \cos(3x) = \frac{1}{2} \quad \text{or} \quad \sin(5x) &= 0\end{aligned}$$

(Case 1).

$$\begin{aligned}\cos(3x) &= \frac{1}{2} \\ 3x &= \pm \frac{\pi}{3} + K \cdot 2\pi \quad \text{where } K \in \mathbb{Z} \\ x &= \pm \frac{\pi}{9} + K \cdot \frac{2\pi}{3}\end{aligned}$$

(Case 2).

$$\begin{aligned}\sin(5x) &= 0 \\ 5x &= M\pi \quad \text{where } M \in \mathbb{Z} \\ x &= M \cdot \frac{\pi}{5}\end{aligned}$$

The general solution of (†) is given by $x = \pm \frac{\pi}{9} + K \cdot \frac{2\pi}{3}$ where $K \in \mathbb{Z}$, or $x = M \cdot \frac{\pi}{5}$ where $M \in \mathbb{Z}$.

Define

$$\begin{aligned}A &= \left\{ x \in \mathbb{R} : x = \frac{\pi}{9} + K \cdot \frac{2\pi}{3} \text{ for some } K \in \mathbb{Z} \right\}, \\ B &= \left\{ x \in \mathbb{R} : x = -\frac{\pi}{9} + L \cdot \frac{2\pi}{3} \text{ for some } L \in \mathbb{Z} \right\}, \\ C &= \left\{ x \in \mathbb{R} : x = M \cdot \frac{\pi}{5} \text{ for some } M \in \mathbb{Z} \right\}\end{aligned}$$

The solution set of (†) is given by $A \cup B \cup C$.

(c) **Solution.**

We proceed to solve the equation (†):

$$\begin{aligned}2 \sin\left(\frac{x}{2}\right) \sin\left(\frac{3x}{2}\right) &= 1 \quad \text{--- (†)} \\ \cos\left(\frac{x}{2} - \frac{3x}{2}\right) - \cos\left(\frac{x}{2} + \frac{3x}{2}\right) &= 1 \\ \cos(-x) - \cos(2x) &= 1 \\ \cos(x) - (2 \cos^2(x) - 1) &= 1 \\ (1 - 2 \cos(x)) \cos(x) &= 0 \\ \cos(x) &= \frac{1}{2} \quad \text{or} \quad \cos(x) = 0\end{aligned}$$

(Case 1).

$$\begin{aligned}\cos(x) &= \frac{1}{2} \\ x &= \pm \frac{\pi}{3} + K \cdot 2\pi \quad \text{where } K \in \mathbb{Z}\end{aligned}$$

(Case 2).

$$\begin{aligned}\cos(x) &= 0 \\ x &= \frac{\pi}{2} + M\pi \quad \text{where } M \in \mathbb{Z}\end{aligned}$$

The general solution of (†) is given by $x = \pm \frac{\pi}{3} + K \cdot 2\pi$ where $K \in \mathbb{Z}$, or $x = \frac{\pi}{2} + M\pi$ where $M \in \mathbb{Z}$.

Define

$$\begin{aligned}A &= \left\{x \in \mathbb{R} : x = \frac{\pi}{3} + K \cdot 2\pi \text{ for some } K \in \mathbb{Z}\right\}, \\ B &= \left\{x \in \mathbb{R} : x = -\frac{\pi}{3} + K \cdot 2\pi \text{ for some } K \in \mathbb{Z}\right\}, \\ C &= \left\{x \in \mathbb{R} : x = \frac{\pi}{2} + M\pi \text{ for some } M \in \mathbb{Z}\right\}\end{aligned}$$

The solution set of (†) is given by $A \cup B \cup C$.

(d) **Solution.**

Note that $6^2 + 8^2 = 100 = 10^2$. Take $\alpha = \arcsin\left(\frac{4}{5}\right)$. Then $10 \sin(\alpha) = 8$, $10 \cos(\alpha) = 10\sqrt{1 - \sin^2(\alpha)} = 6$.

We proceed to solve the equation (†):

$$\begin{aligned}6 \sin(x) + 8 \cos(x) &= 5 \quad \text{--- (†)} \\ 10 \sin(x) \cos(\alpha) + 10 \cos(x) \sin(\alpha) &= 5 \\ 10 \sin(x + \alpha) &= 5 \\ \sin(x + \alpha) &= \frac{1}{2} \\ x + \alpha &= (-1)^N \cdot \frac{\pi}{6} + N\pi \quad \text{where } N \in \mathbb{Z} \\ x &= \alpha + (-1)^N \cdot \frac{\pi}{6} + N\pi\end{aligned}$$

The general solution of (†) is given by $x = \alpha + (-1)^N \cdot \frac{\pi}{6} + N\pi$ where $N \in \mathbb{Z}$.

The solution set of (†) is given by $\left\{x \in \mathbb{R} : x = \alpha + (-1)^N \cdot \frac{\pi}{6} + N\pi \text{ for some } N \in \mathbb{Z}\right\}$.

(e) **Solution.**

We proceed to solve the equation (†):

$$\begin{aligned}\tan(3\sqrt{x}) &= 1 \quad \text{---} \quad (\dagger) \\ 3\sqrt{x} &= \frac{\pi}{4} + N \cdot \pi \quad \text{where } N \in \mathbb{N} \\ x &= \left(\frac{1}{4} + N\right)^2 \cdot \frac{\pi^2}{9}\end{aligned}$$

The general solution of (†) is given by $x = \left(\frac{1}{4} + N\right)^2 \cdot \frac{\pi^2}{9}$, where N is a natural number.

The solution set of (†) is given by $\left\{x \in \mathbb{R} : x = \left(\frac{1}{4} + N\right)^2 \cdot \frac{\pi^2}{9} \text{ for some } N \in \mathbb{N}\right\}$.

(f) **Solution.**

We proceed to solve the equation (†):

$$\begin{aligned}\cos\left(\frac{1}{2x}\right) &= 1 \quad \text{---} \quad (\dagger) \\ \frac{1}{2x} &= 2N \cdot \pi \quad \text{where } N \in \mathbb{Z} \setminus \{0\} \\ x &= \frac{1}{4N \cdot \pi}\end{aligned}$$

The general solution of (†) is given by $x = \frac{1}{4N \cdot \pi}$ where $N \in \mathbb{Z} \setminus \{0\}$.

The solution set of (†) is given by $\left\{x \in \mathbb{R} : x = \frac{1}{4N \cdot \pi} \text{ for some } N \in \mathbb{Z} \setminus \{0\}\right\}$.

6. **Solution.**

(a) Let $z = \frac{2(\sqrt{3} - i)}{\sqrt{3} + i}$.

Note that $z = \frac{2(\sqrt{3} - i)}{\sqrt{3} + i} = 2 \cdot \frac{(\sqrt{3} - i)/2}{(\sqrt{3} + i)/2} = 2 \cdot \frac{\cos(-\pi/6) + i \sin(-\pi/6)}{\cos(\pi/6) + i \sin(\pi/6)} = 2 \cdot \left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)\right)$.

(b) i. We have $|z|^6 = 2^6$. Note that $2020 = 6 \cdot 336 + 4$. Then

$$\begin{aligned}z^{2020} &= z^{6 \cdot 336 + 4} = (z^6)^{336} \cdot z^4 = (2^6)^{336} \cdot 2^4 \cdot \cos\left(-4 \cdot \frac{\pi}{3}\right) + i \sin\left(-4 \cdot \frac{\pi}{3}\right) = 2^{2020} \left(\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right) \\ &= -2^{2019} + 2^{2019} \cdot \sqrt{3}i\end{aligned}$$

ii. The square roots of z are $\pm\sqrt{2} \cdot \left(\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right)$ respectively.

Expressed in standard form, they are $\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$, $-\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$ respectively.

7. (a) **Solution.**

We proceed to solve the equation (★):

$$\begin{aligned}z^5 - 32i &= 0 \quad \text{---} \quad (\star) \\ z^5 &= 32i \\ \left(\frac{z}{2i}\right)^5 &= 1 \\ \frac{z}{2i} &= \omega^N, \quad \text{where } N = 0, 1, 2, 3, 4.\end{aligned}$$

Here ω stands for the number $\cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right)$.

So the general solution of the equation (\star) is given by $z = 2(\cos(\frac{\pi}{2} + N \cdot \frac{2\pi}{5}) + i \sin(\frac{\pi}{2} + N \cdot \frac{2\pi}{5}))$, where $N = 0, 1, 2, 3, 4$.

(b) **Answer.**

$$z = 2(\cos(\frac{\pi}{10}) + i \sin(\frac{\pi}{10})) \text{ or } z = 2(\cos(-\frac{3\pi}{10}) + i \sin(-\frac{3\pi}{10})) \text{ or } z = 2(\cos(-\frac{7\pi}{10}) + i \sin(-\frac{7\pi}{10})).$$

8. (a) *Hint.* You need De Moivre's Theorem. Also recall that for any $\eta \in \mathbb{C}$, we have $2\operatorname{Re}(\eta) = \eta + \bar{\eta}$ and $2i\operatorname{Im}(\eta) = \eta - \bar{\eta}$.

(b) *Hint.* You need the Binomial Theorem.

9. **Answer.**

(a) (I) $x \in A$

(II) There exists some

(III) $x = 16m^6$

(IV) $2, m \in \mathbb{Z}$

(V) $x = 2(8m^6) = 2(2m^2)^3 = 2n^3$

(VI) $x \in B$

(b) (I) $x_0 = 2 \cdot 1^3$

(II) 1

(III) $x_0 \in B$

(IV) Suppose it were true that $x_0 \in A$.

(V) there would exist some $m \in \mathbb{Z}$

(VI) $2 \cdot (4m^6) = 8m^6 = 1$

(VII) $4m^6 \in \mathbb{Z}$

(VIII) divisible

10. **Solution.**

Let $A = \{x \mid x = r^6 \text{ for some } r \in \mathbb{Q}\}$, $B = \{x \mid x = r^2 \text{ for some } r \in \mathbb{Q}\}$.

(a) Pick any $x \in A$.

There exists some $r \in \mathbb{Q}$ such that $x = r^6$.

Take $s = r^3$. Note that $s \in \mathbb{Q}$. We have $x = (r^3)^2 = s^2$.

Hence $x \in B$.

It follows that $A \subset B$.

(b) Let $x_0 = 4$.

Note that $x_0 = 2^2$ and $2 \in \mathbb{Q}$. Then $x_0 \in B$.

We verify that $x_0 \notin A$:

- Suppose it were true that $x_0 \in A$.

Then there would exist some $r \in \mathbb{Q}$ such that $x_0 = r^6$.

Now $4 = r^6$.

Since $r \in \mathbb{R}$, we would have $r = \sqrt[3]{2}$ or $r = -\sqrt[3]{2}$.

But $\sqrt[3]{2}, -\sqrt[3]{2} \notin \mathbb{Q}$. Contradiction arises.

Therefore the assumption that $x_0 \in A$ is false. We have $x_0 \notin A$.

Hence $B \not\subset A$.

11. **Solution.**

Let $C = \{\zeta \in \mathbb{C} : |\zeta - 1| \leq 1\}$, $D = \{\zeta \in \mathbb{C} : |\zeta| \leq 2\}$.

(a) Pick any $\zeta \in C$.

We have $|\zeta - 1| \leq 1$ (by the definition of C).

By the Triangle Inequality, we have $|\zeta| - 1 \leq |\zeta - 1| \leq 1$.

Then $|\zeta| \leq 2$. Therefore, we have $\zeta \in D$ (by the definition of D).

It follows that $C \subset D$.

(b) Take $\zeta_0 = -1$. We have $\zeta_0 \in \mathbb{C}$.

Note that $|\zeta_0| = 1 \leq 2$. Then $\zeta_0 \in D$.

We verify that $\zeta_0 \notin C$:

- We have $|\zeta_0 - 1| = |-1 - 1| = 2 > 1$. Then $\zeta_0 \notin C$.

It follows that $D \not\subset C$.

12. **Answer.**

(a) (I) $A \cup B \subset B$

(II) it were true that $A \setminus B \neq \emptyset$

(III) $x_0 \in A \setminus B$

(IV) $x_0 \in A$

(V) $x_0 \in B$

(VI) $x_0 \in A \cup B$

(VII) $x_0 \in B$

(VIII) Contradiction arises

(b) —

13. (a) **Solution.**

Let A, B, C, D be sets. Suppose $A \subset C$ and $B \subset D$.

Pick any object x .

Suppose $x \in A \cup B$. Then $x \in A$ or $x \in B$.

(Case 1). Suppose $x \in A$. Then, since $A \subset C$, we have $x \in C$. Therefore $x \in C$ or $x \in D$. Hence $x \in C \cup D$.

(Case 2). Suppose $x \notin A$. Then $x \in B$. Therefore, since $B \subset D$, we have $x \in D$. Then $x \in C$ or $x \in D$. Hence $x \in C \cup D$.

Hence in any case, we have $x \in C \cup D$.

It follows that $A \cup B \subset C \cup D$.

(b) —

(c) —

14. —

15. **Answer.**

(a) False.

(b) True.

(c) False.

(d) False.

(e) True.

16. **Solution.**

Consider the predicate ' $x \notin x$ ', which we denote by $P(x)$.

(a) Denote by R the object $\{x \mid P(x)\}$. Suppose it were true that R was a set.

i. Suppose it were true that the object R is an element of the set R . Then $R \in R$. By the definition of R , $P(R)$ is a true statement. Then $R \notin R$. We have $R \in R$ and $R \notin R$. Contradiction arises.

Hence the object R is not an element of the set R .

ii. Suppose it were true that the object R is not an element of the set R . Then $R \notin R$. Therefore, $P(R)$ is a true statement. Hence, by the definition of R , $R \in R$. We have $R \notin R$ and $R \in R$. Contradiction arises.

Hence the object R is an element of the set R .

Remark. Hence the object R is not a set in the first place. The failure of the construction $\{x \mid x \notin x\}$ in giving a set is known as **Russell's Paradox**.

(b) Let A be a set. Denote by B the object $\{x \in A : P(x)\}$.

Suppose B were an element of A . Then $B \in A$. We have $B \in B$ or $B \notin B$.

• (Case 1). Suppose $B \in B$. Then, by the definition of B , $P(B)$ is a true statement. Therefore $B \notin B$. We have $B \in B$ and $B \notin B$. Contradiction arises.

• (Case 2). Suppose $B \notin B$. Then $P(B)$ is a true statement, Therefore, by the definition of B , $B \in B$. We have $B \notin B$ and $B \in B$. Contradiction arises.

Hence, in every case, contradiction arises. B is not an element of A in the first place.