

1. **Solution.**

(a) Let $n \in \mathbb{N}$, and $k \in \mathbb{Z}$.

i. • (Case 1.) Suppose $0 \leq k \leq n$.

$$\text{Then } \binom{n}{k} = \frac{n!}{k! \cdot [(n-k)!]} = \frac{n!}{[n - (n-k)]! \cdot [(n-k)!]} = \frac{n!}{[(n-k)!] \cdot [n - (n-k)]!} = \binom{n}{n-k}.$$

• (Case 2.) Suppose $k < 0$. Then $n - k > n$. Therefore $\binom{n}{k} = 0 = \binom{n}{n-k}$.

• (Case 3.) Suppose $k > n$. Then $n - k < 0$. Therefore $\binom{n}{k} = 0 = \binom{n}{n-k}$.

ii. • (Case 1.) Suppose $0 \leq k \leq n - 1$. Then

$$\begin{aligned} \binom{n}{k} + \binom{n}{k+1} &= \frac{n!}{k! \cdot (n-k)!} + \frac{n!}{(k+1)! \cdot (n-k-1)!} \\ &= \frac{n!}{k! \cdot (n-k-1)!} \cdot \left(\frac{1}{n-k} + \frac{1}{k+1} \right) \\ &= \frac{n!}{k! \cdot (n-k-1)!} \cdot \frac{n+1}{(n-k)(k+1)} \\ &= \frac{(n+1)!}{(k+1)! \cdot (n-k)!} = \frac{(n+1)!}{(k+1)! \cdot [(n+1) - (k+1)]!} = \binom{n+1}{k+1}. \end{aligned}$$

• (Case 2.) Suppose $k < -1$. Then $k+1 < 0$.

$$\binom{n}{k} = \binom{n}{k+1} = 0 = \binom{n+1}{k+1}. \text{ Therefore } \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

• (Case 3.) Suppose $k = -1$. Then $k+1 = 0$.

$$\binom{n}{k} = 0 \text{ and } \binom{n}{k+1} = 1 = \binom{n+1}{k+1}. \text{ Therefore } \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

• (Case 4.) Suppose $k \geq n$. Then $n - k \leq 0$. Also, $n - k - 1 \leq 0$

$$\begin{aligned} \binom{n}{k} + \binom{n}{k+1} &= \binom{n}{n-k} + \binom{n}{n-k-1} \\ &= \binom{n}{(n-k-1)+1} + \binom{n}{n-k-1} \\ &= \binom{n+1}{(n-k-1)+1} = \binom{n+1}{(n+1) - (k+1)} = \binom{n+1}{k+1} \end{aligned}$$

(b) i. Suppose $n, m \in \mathbb{N}$.

$$\text{For each } j = 1, 2, \dots, m, \text{ we have } \binom{n+j+1}{j} - \binom{n+j}{j-1} = \binom{n+j}{j}.$$

Then

$$\begin{aligned} \sum_{j=0}^m \binom{n+j}{j} &= \binom{n}{0} + \sum_{j=1}^m \binom{n+j}{j} \\ &= \binom{n}{0} + \sum_{j=1}^m \left(\binom{n+j+1}{j} - \binom{n+j}{j-1} \right) \\ &= \binom{n}{0} + \left(\binom{n+m+1}{m} - \binom{n+1}{1-1} \right) \\ &= \binom{n+m+1}{m} \end{aligned}$$

ii. Suppose $n, m \in \mathbb{N}$.

$$\text{For each } j = 1, 2, \dots, m, \text{ we have } \binom{n+j+1}{n+1} - \binom{n+j}{n+1} = \binom{n+j}{n}$$

Then

$$\begin{aligned}
\sum_{j=0}^m \binom{n+j}{n} &= \binom{n}{n} + \sum_{j=1}^m \binom{n+j}{n} \\
&= \binom{n}{n} + \sum_{j=1}^m \left(\binom{n+j+1}{n+1} - \binom{n+j}{n+1} \right) \\
&= \binom{n}{n} + \left(\binom{n+m+1}{n+1} - \binom{n+1}{n+1} \right) \\
&= \binom{n+m+1}{n+1}
\end{aligned}$$

2. Solution.

Let n be a positive integer.

(a) Suppose r is an integer amongst $0, 1, \dots, n$.

$$\text{Then } \binom{n+1}{r+1} / \binom{n+1}{r} = \frac{(n+1)!}{[(r+1)! \{(n+1)-r-1\}]} \cdot \frac{(r!) \{(n+1)-r\}!}{(n+1)!} = \dots = \frac{n+1-r}{r+1}.$$

(b) i.

$$\begin{aligned}
\sum_{k=0}^n (k+1) \cdot \binom{n+1}{k+1} / \binom{n+1}{k} &= \sum_{k=0}^n (k+1) \cdot \frac{n+1-k}{k+1} = \sum_{k=0}^n (n+1-k) \\
&= \sum_{k=0}^n (n+1) - \sum_{k=0}^n k \\
&= (n+1)^2 - \frac{(n+1)n}{2} = \frac{n^2 + 3n + 2}{2}
\end{aligned}$$

ii.

$$\begin{aligned}
\prod_{k=0}^n \left(\binom{n+1}{k+1} + \binom{n+1}{k} \right) &= \prod_{k=0}^n \left[\left(\binom{n+1}{k+1} / \binom{n+1}{k} + 1 \right) \cdot \binom{n+1}{k} \right] \\
&= \prod_{k=0}^n \left[\left(\frac{n+1-k}{k+1} + 1 \right) \cdot \binom{n+1}{k} \right] \\
&= \prod_{k=0}^n \left[\frac{n+2}{k+1} \cdot \binom{n+1}{k} \right] \\
&= \left[\prod_{k=0}^n (n+2) \right] \left[\prod_{k=0}^n \frac{1}{k+1} \right] \cdot \left(\prod_{k=0}^n \binom{n+1}{k} \right) \\
&= \frac{(n+2)^{n+1}}{[(n+1)!]} \cdot \left(\prod_{k=0}^n \binom{n+1}{k} \right)
\end{aligned}$$

3. Solution.

Denote by $P(n)$ the following proposition:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{k}x^k + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n \text{ as polynomials.}$$

- Note that $(1+x)^0 = 1 = \binom{0}{0}$ as polynomials. Hence $P(0)$ is true.
- Let $m \in \mathbb{N}$. Suppose $P(m)$ is true. Then

$$(1+x)^m = \binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \dots + \binom{m}{k}x^k + \dots + \binom{m}{m-1}x^{m-1} + \binom{m}{m}x^m$$

as polynomials.

We verify that $P(m+1)$ is true:

As polynomials,

$$\begin{aligned}
(1+x)^{m+1} &= (1+x)(1+x)^m \\
&= (1+x)\left(\binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \cdots \right. \\
&\quad \left. + \binom{m}{k}x^k + \cdots + \binom{m}{m-1}x^{m-1} + \binom{m}{m}x^m\right) \\
&= \left(\binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \cdots + \binom{m}{k}x^k + \binom{m}{k+1}x^{k+1} + \cdots + \binom{m}{m}x^m\right) \\
&\quad + \left(\binom{m}{0}x + \binom{m}{1}x^2 + \cdots \right. \\
&\quad \left. + \binom{m}{k-1}x^k + \binom{m}{k}x^{k+1} + \cdots + \binom{m}{m-1}x^m + \binom{m}{m}x^{m+1}\right) \\
&= \binom{m}{0} + \left(\binom{m}{0} + \binom{m}{1}\right)x + \left(\binom{m}{1} + \binom{m}{2}\right)x^2 + \cdots + \left(\binom{m}{k-1} + \binom{m}{k}\right)x^k \\
&\quad + \left(\binom{m}{k} + \binom{m}{k+1}\right)x^{k+1} + \cdots + \left(\binom{m}{m-1} + \binom{m}{m}\right)x^m + \binom{m}{m}x^{m+1} \\
&= \binom{m+1}{0} + \binom{m+1}{1}x + \binom{m+1}{2}x^2 + \cdots + \binom{m+1}{k}x^k + \binom{m+1}{k+1}x^{k+1} \\
&\quad + \cdots + \binom{m+1}{m}x^m + \binom{m+1}{m+1}x^{m+1}
\end{aligned}$$

It follows that $P(m+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

4. (a) **Solution.**

Let n be a positive integer, and $f(x)$ be the polynomial $f(x) = (1+x)^n$.

Note that $f(x) = \sum_{k=0}^n \binom{n}{k} x^k$ as polynomials.

- i. $\sum_{k=0}^n \binom{n}{k} = \sum_{k=0}^n \binom{n}{k} \cdot 1^k = f(1) = (1+1)^n = 2^n$.
- ii. $\sum_{k=0}^n (-1)^k \binom{n}{k} = f(-1) = (1-1)^n = 0$.
- iii. $\sum_{k=0}^n \frac{1}{2^k} \binom{n}{k} = f\left(\frac{1}{2}\right) = \left(1 + \frac{1}{2}\right)^n = \frac{3^n}{2^n}$.
- iv. $\sum_{k=0}^n \frac{(-1)^k \cdot 3^{k-1}}{5^{k+1}} \binom{n}{k} = \frac{1}{15} \sum_{k=0}^n \frac{(-1)^k \cdot 3^k}{5^k} \binom{n}{k} = \frac{1}{15} f\left(-\frac{3}{5}\right) = \frac{1}{15} \left(1 - \frac{3}{5}\right)^n = \frac{2^n}{15 \cdot 5^n}$.

(b) **Solution.**

Let m be a positive integer. Then $2m$ is a positive integer.

Let $g(x)$ be the polynomial $g(x) = (1+x)^{2m}$.

Note that $g(x) = \sum_{k=0}^{2m} \binom{2m}{k} x^k$ as polynomials.

- i. $\sum_{k=0}^{2m} \binom{2m}{k} = g(1) = 2^{2m}$.
- ii. $\sum_{k=0}^{2m} (-1)^k \binom{2m}{k} = g(-1) = 0$.
- iii.

$$\begin{aligned}
\sum_{k=0}^m \binom{2m}{2k} &= \sum_{j=0}^{2m} \frac{1}{2} \left(\binom{2m}{j} + (-1)^j \binom{2m}{j} \right) \\
&= \frac{1}{2} \left(\sum_{j=0}^{2m} \binom{2m}{j} + \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} \right) = \frac{1}{2} (2^{2m} + 0) = 2^{2m-1}
\end{aligned}$$

iv.

$$\begin{aligned} \sum_{k=0}^{m-1} \binom{2m}{2k+1} &= \sum_{j=0}^{2m} \frac{1}{2} \left(\binom{2m}{j} - (-1)^j \binom{2m}{j} \right) \\ &= \frac{1}{2} \left(\sum_{j=0}^{2m} \binom{2m}{j} - \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} \right) = \frac{1}{2} (2^{2m} - 0) = 2^{2m-1} \end{aligned}$$

(c) *Hint.* Note that $4p$ is a positive integer. Define the polynomial $h(z) = (1+z)^{4p}$.

The results in the previous parts give

$$\sum_{k=0}^{4p} \binom{4p}{k} = 2^{4p}, \quad \sum_{k=0}^{4p} (-1)^k \binom{4p}{k} = 0, \quad \sum_{k=0}^{2p} \binom{4p}{2k} = 2^{4p-1}, \quad \sum_{k=0}^{2p-1} \binom{4p}{2k+1} = 2^{4p-1}.$$

Further obtain

$$(-1)^p \cdot 2^{2p} = h(i) = \sum_{j=0}^{2p} (-1)^j \binom{4p}{2j} + i \sum_{j=0}^{2p-1} (-1)^j \binom{4p}{2j+1}$$

Make use of these above to the answers for the respective parts.

Answer.

- i. $(-1)^p \cdot 2^{2p-1}$.
- ii. 0.
- iii. $2^{4p-2} + (-1)^p \cdot 2^{2p-1}$.
- iv. $2^{4p-2} - (-1)^p \cdot 2^{2p-1}$.
- v. 2^{4p-2} .
- vi. 2^{4p-2} .

5. Solution.

(a) Let $n \in \mathbb{N} \setminus \{0\}$, and $k \in \mathbb{Z}$.

- (Case 1.) Suppose $0 < k \leq n$. Then

$$k \cdot \binom{n}{k} = k \cdot \frac{n!}{k! \cdot (n-k)!} = \frac{n!}{(k-1)! \cdot (n-k)!} = n \cdot \frac{(n-1)!}{(k-1)! \cdot [(n-1) - (k-1)]!} = n \cdot \binom{n-1}{k-1}.$$

- (Case 2.) Suppose $k \leq 0$ or $k > n$. Then $k \cdot \binom{n}{k} = 0 = n \cdot \binom{n-1}{k-1}$.

Hence in any case, $k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$.

(b) Let n be a positive integer.

$$\text{i. } \sum_{k=0}^n k \binom{n}{k} = \sum_{k=1}^n k \binom{n}{k} = \sum_{k=1}^n n \binom{n-1}{k-1} = n \sum_{k=1}^n \binom{n-1}{k-1} = n \sum_{j=0}^{n-1} \binom{n-1}{j} = n \cdot 2^{n-1}.$$

Alternative argument. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = (1+x)^n$ for any $x \in \mathbb{R}$. f is smooth on \mathbb{R} , and for any $x \in \mathbb{R}$, we have

$$n(1+x)^{n-1} = f'(x) = \sum_{k=1}^n k \binom{n}{k} x^{k-1}.$$

Then $\sum_{k=0}^n k \binom{n}{k} = f'(1) = n \cdot 2^{n-1}$.

ii. We have $n \leq 1$ or $n \geq 2$.

- (Case 1.) Suppose $n = 1$. Then We have $\sum_{k=0}^n (-1)^{k+1} k \binom{n}{k} = \sum_{k=0}^1 (-1)^{k+1} k \binom{n}{k} = (-1)^{1+1} \cdot 1 \cdot \binom{1}{1} = 1$.

- (Case 2.) Suppose $n \geq 2$. Then we have

$$\begin{aligned}
\sum_{k=0}^n (-1)^{k+1} k \binom{n}{k} &= \sum_{k=1}^n (-1)^{k+1} k \binom{n}{k} \\
&= \sum_{k=1}^n (-1)^{k+1} n \binom{n-1}{k-1} \\
&= n \sum_{k=1}^n (-1)^{k+1} \binom{n-1}{k-1} \\
&= n \sum_{j=0}^{n-1} (-1)^{j+2} \binom{n-1}{j} \\
&= n \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} = n \cdot 0 = 0
\end{aligned}$$

iii. We have $n \geq 2$ or $n \leq 1$.

- (Case 1.) Suppose $n \geq 2$. Then

$$\begin{aligned}
\sum_{k=0}^n k(k-1) \binom{n}{k} &= \sum_{k=2}^n k(k-1) \binom{n}{k} \\
&= \sum_{k=2}^n n(n-1) \binom{n-2}{k-2} \\
&= n(n-1) \sum_{k=2}^n \binom{n-2}{k-2} \\
&= n(n-1) \sum_{j=0}^{n-2} \binom{n-2}{j} = n(n-1) \cdot 2^{n-2}
\end{aligned}$$

- (Case 2.) Suppose $n \leq 1$. Then $\sum_{k=0}^n k(k-1) \binom{n}{k} = 0 = n(n-1) \cdot 2^{n-2}$.

Hence in any case, $\sum_{k=0}^n k(k-1) \binom{n}{k} = n(n-1) \cdot 2^{n-2}$.

iv.

$$\begin{aligned}
\sum_{k=0}^n k^2 \binom{n}{k} &= \sum_{k=0}^n [k(k-1) + k] \binom{n}{k} \\
&= \sum_{k=0}^n k(k-1) \binom{n}{k} + \sum_{k=0}^n k \binom{n}{k} \\
&= n(n-1) \cdot 2^{n-2} + n \cdot 2^{n-1} = n(n+1) \cdot 2^{n-2}
\end{aligned}$$

(c) Let m be a positive integer. Note that $m+1 > 1$.

For each non-negative integer k , we have $(k+1) \cdot \binom{m+1}{k+1} = (m+1) \cdot \binom{m}{k}$.

Therefore

$$\frac{1}{k+1} \cdot \binom{m}{k} = \frac{1}{m+1} \cdot \binom{m+1}{k+1}.$$

i.

$$\begin{aligned}
\sum_{k=0}^m \frac{1}{k+1} \binom{m}{k} &= \sum_{k=0}^m \frac{1}{m+1} \binom{m+1}{k+1} \\
&= \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k+1} \\
&= \frac{1}{m+1} \sum_{j=1}^{m+1} \binom{m+1}{j} \\
&= \frac{1}{m+1} \left(\sum_{j=0}^{m+1} \binom{m+1}{j} - 1 \right) = \frac{2^{m+1} - 1}{m+1}
\end{aligned}$$

Alternative argument. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = (1+x)^n$ for any $x \in \mathbb{R}$. f is continuous on \mathbb{R} , and for any $x \in \mathbb{R}$, we have

$$\frac{(1+x)^{n+1} - 1}{n+1} = \int_0^x f(t) dt = \sum_{k=0}^n \frac{1}{k+1} \binom{n}{k} x^{k+1}.$$

Then $\sum_{k=0}^n \frac{1}{k+1} \binom{n}{k} = \int_0^1 f(t) dt = \frac{2^{n+1} - 1}{n+1}$.

ii.

$$\begin{aligned}
\sum_{k=0}^m \frac{(-1)^k}{k+1} \binom{m}{k} &= \sum_{k=0}^m \frac{(-1)^k}{m+1} \binom{m+1}{k+1} \\
&= \frac{1}{m+1} \sum_{k=0}^m (-1)^k \binom{m+1}{k+1} \\
&= \frac{1}{m+1} \sum_{j=1}^{m+1} (-1)^{j-1} \binom{m+1}{j} \\
&= \frac{1}{m+1} \left(- \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} + 1 \right) = \frac{1}{m+1}
\end{aligned}$$

iii.

$$\begin{aligned}
\sum_{k=0}^m \frac{1}{(k+2)(k+1)} \binom{m}{k} &= \sum_{k=0}^m \frac{1}{m+1} \cdot \frac{1}{k+2} \binom{m+1}{k+1} \\
&= \sum_{k=0}^m \frac{1}{(m+2)(m+1)} \binom{m+2}{k+2} \\
&= \frac{1}{(m+2)(m+1)} \sum_{k=0}^m \binom{m+2}{k+2} \\
&= \frac{1}{(m+2)(m+1)} \sum_{j=2}^{m+2} \binom{m+2}{j} \\
&= \frac{1}{(m+2)(m+1)} \left[\sum_{j=0}^{m+2} \binom{m+2}{j} - 1 - (m+2) \right] = \frac{2^{m+2} - m - 3}{(m+2)(m+1)}
\end{aligned}$$

6. Solution.

Let α be a complex number. Suppose $0 < |\alpha| < 1$.

Define $\beta = \frac{1}{|\alpha|} - 1$. We have $\beta > 0$.

(a) Suppose $n \geq 2$. Then, by Bernoulli's Inequality, $(1+\beta)^n \geq 1+n\beta \geq n\beta$. Therefore $|\alpha|^n = \frac{1}{(1+\beta)^n} \leq \frac{1}{n\beta}$.

(b) Suppose $n \geq 3$. Then $(1+\beta)^n = 1+n\beta + \frac{n(n-1)}{2}\beta^2 + \underbrace{\dots \dots \dots}_{\text{finitely many non-negative terms}} \geq \frac{n(n-1)}{2}\beta^2$.

Therefore $n|\alpha|^n = \frac{n}{(1+\beta)^n} \leq \frac{2}{(n-1)\beta^2} = \frac{2}{[n/2 + (n/2 - 1)]\beta^2} \leq \frac{2}{(n/2)\beta^2} = \frac{4}{n\beta^2}$.

(c) Suppose $n \geq 4$. Then

$$(1 + \beta)^n = 1 + n\beta + \frac{n(n-1)}{2}\beta^2 + \frac{n(n-1)(n-2)}{6}\beta^3 + \underbrace{\dots \dots \dots}_{\text{finitely many non-negative terms}} \geq \frac{n(n-1)(n-2)}{6}\beta^3.$$

Therefore

$$n^2|\alpha|^n = \frac{n^2}{(1 + \beta)^n} \leq \frac{6n}{(n-1)(n-2)\beta^3} = \frac{6n}{[n/2 + (n/2 - 1)][n/3 + (2n/3 - 2)]\beta^3} \leq \frac{6n}{(n/2)(n/3)\beta^3} = \frac{36}{n\beta^3}.$$

(d) *Hint.* Generalize what you see in part (b) and part (c).

7. —

8. **Answer.**

(a) —

(b) i. $f'(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^{-2}e^{-1/x} & \text{if } x > 0 \end{cases}.$

ii. —
iii. —

(c) i. *Hint.* Apply mathematical induction to the proposition $S(n)$ below:

- There exists some polynomial function P_n such that

$$f^{(n)}(x) = \begin{cases} 0 & \text{if } x < 0 \\ P_n(1/x)e^{-1/x} & \text{if } x > 0 \end{cases}.$$

ii. *Hint.* Apply mathematical induction to the proposition $T(n)$ below:

- f is n -times differentiable at 0 and $f^{(n)}(0) = 0$.

Be careful: that f is n -times differentiable at 0 for each specific n is something which needs being argued for. It should not be taken for granted, even though it is apparent by the result of the previous part that f is smooth at every point of \mathbb{R} other than 0.

Remark. The Taylor series $T_{f,0}(x)$ of the function f about the point 0 is given by the expression $\sum_{n=0}^{\infty} 0 \cdot x^n$.

But f is not the zero function; in fact, for any $\delta > 0$, we have $f(\delta/2) \neq 0$. Therefore f fails to be equal to the constant zero function on $(0, \delta)$,

9. **Solution.**

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Suppose that $f(x + y) = f(x) + f(y)$ for any $x, y \in \mathbb{R}$.

(a) We have $f(0) = f(0 + 0) = f(0) + f(0)$. Then $f(0) = 0$.

(b) Let $x \in \mathbb{R}$. Note that $-x \in \mathbb{R}$.

We have $f(x) + f(-x) = f(x + (-x)) = f(0) = 0$. Then $f(-x) = -f(x)$.

(c) Denote by $P(n)$ the proposition $f(n) = nf(1)$.

- We have $f(0) = 0 = 0 \cdot f(1)$. Then $P(1)$ is true.
- Let $k \in \mathbb{N}$. Suppose $P(k)$ is true. Then $f(k) = kf(1)$.

We verify that $P(k + 1)$ is true:

$$\text{We have } f(k + 1) = f(k) + f(1) = kf(1) + f(1) = (k + 1)f(1).$$

Hence $P(k + 1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

(d) Let $m \in \mathbb{Z}$.

- Suppose $m \geq 0$. Then $m \in \mathbb{N}$. We have $f(m) = mf(1)$.
- Suppose $m < 0$. Then $-m > 0$.

$$\text{We have } -f(m) = f(-m) = (-m) \cdot f(1) = -mf(1).$$

Then $f(m) = mf(1)$.

Hence, in any case, we have $f(m) = mf(1)$.

(e) Let $r \in \mathbb{Q}$. There exist some $m, n \in \mathbb{Z}$ such that $n \neq 0$ and $m = nr$.

$$\text{Now } mf(1) = f(m) = f(nr) = nf(r).$$

$$\text{Then } f(r) = \frac{m}{n}f(1) = rf(1).$$

(f) Now further suppose that f is continuous on \mathbb{R} .

Let $u \in \mathbb{R}$. There exists some infinite sequence of rational numbers $\{s_n\}_{n=0}^{\infty}$ such that $\lim_{n \rightarrow \infty} s_n = u$.

For each $n \in \mathbb{N}$, we have $f(s_n) = f(1)s_n$.

Then, by continuity, we have $f(u) = f\left(\lim_{n \rightarrow \infty} s_n\right) = \lim_{n \rightarrow \infty} f(s_n) = \lim_{n \rightarrow \infty} f(1)s_n = f(1)u$.