

MATH1050 Exercise 4 (Answers and solution)

1. **Solution.**

(a) Denote by $P(n)$ the proposition

$$1 \cdot 2 + 2 \cdot 5 + 3 \cdot 8 + \cdots + n(3n - 1) = n^2(n + 1).$$

- Note that $1 \cdot 2 = 2 = 1^2(1 + 1)$. Then $P(1)$ is true.
- Let k be a positive integer. Suppose $P(k)$ is true. Then

$$1 \cdot 2 + 2 \cdot 5 + 3 \cdot 8 + \cdots + k(3k - 1) = k^2(k + 1).$$

We verify that $P(k + 1)$ is true:

We have

$$\begin{aligned} & 1 \cdot 2 + 2 \cdot 5 + 3 \cdot 8 + \cdots + k(3k - 1) + (k + 1)[3(k + 1) - 1] \\ &= k^2(k + 1) + (k + 1)(3k + 2) = (k + 1)[k^2 + (3k + 2)] = \cdots = (k + 1)^2[(k + 1) + 1] \end{aligned}$$

Hence $P(k + 1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any positive integer n .

(b) Denote by $P(n)$ the proposition

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}.$$

- We have $1 \geq \sqrt{1}$. Hence $P(1)$ is true.
- Let k be a positive integer. Suppose $P(k)$ is true. Then $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} \geq \sqrt{k}$.

We verify that $P(k + 1)$ is true:

$$\begin{aligned} 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} &\geq \sqrt{k} + \frac{1}{\sqrt{k+1}} \\ &= \frac{\sqrt{k} \cdot \sqrt{k+1} + 1}{\sqrt{k+1}} \\ &\geq \frac{\sqrt{k} \cdot \sqrt{k+1}}{\sqrt{k+1}} \\ &= \sqrt{k+1} \end{aligned}$$

Hence $P(k + 1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true whenever n is a positive integer.

(c) Denote by $P(n)$ the proposition

$$n^2 < 2^{n-1}.$$

- We have $7^2 = 49 < 64 = 2^7 - 1$. Then $P(7)$ is true.
- Let k be an integer greater than 6. Suppose $P(k)$ is true. Then $k^2 < 2^{k-1}$. Therefore $2^{k-1} > 2^k$.
We have

$$\begin{aligned} 2^{(k+1)-1} - (k + 1)^2 &= 2^k - (k^2 + 2k + 1) \\ &\geq 2^k - k^2 - 2k - k \\ &= 2^k - k^2 - 3k \\ &\geq 2^k k^2 - k \cdot k = 2(2^{k-1} - k^2) > \geq 0. \end{aligned}$$

Then $(k + 1)^2 < 2^{(k+1)-1}$. Hence $P(k + 1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true whenever k is an integer greater than 6.

(d) Denote by $P(n)$ the proposition that $n(n^2 + 2)$ is divisible by 3.

- We have $0 \cdot (0^2 + 2) = 0 = 3 \cdot 0$ and $0 \in \mathbb{Z}$. Hence $0 \cdot (0^2 + 2)$ is divisible by 3.
Then $P(0)$ is true.
- Let k be a positive integer. Suppose $P(k)$ is true. Then $k(k^2 + 2)$ is divisible by 3. Therefore there exists some $q \in \mathbb{Z}$ such that $k(k^2 + 2) = 3q$.

We verify that $P(k + 1)$ is true:

We have

$$(k + 1)[(k + 1)^2 + 2] = k^3 + 3k^2 + 5k + 3 = k(k^2 + 2) + 3k^2 + 3 = 3q + 3k^2 + 3 = 3(q + k^2 + 1).$$

Note that $q + k^2 + 1 \in \mathbb{Z}$. Then $(k + 1)[(k + 1)^2 + 2]$ is divisible by 3.

Hence $P(k + 1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

(e) Denote by $P(n)$ the proposition that $7^n(3n+1) - 1$ is divisible by 9.

- We have $7^0(3 \cdot 0 + 1) - 1 = 0$. 0 is divisible by 9. Then $P(0)$ is true.
- Let $k \in \mathbb{N}$. Suppose $P(k)$ is true. Then $7^k(3k+1) - 1$ is divisible by 9. Therefore there exists some $q \in \mathbb{Z}$ such that $7^k(3k+1) - 1 = 9q$.
We verify that $P(k+1)$ is true:

$$\begin{aligned} 7^{k+1}[3(k+1)+1] - 1 &= 7 \cdot 7^k[(3k+1)+3] - 1 \\ &= 7 \cdot 7^k(3k+1) + 3 \cdot 7^{k+1} - 1 \\ &= 7 \cdot [7^k(3k+1) - 1] + 3(7^{k+1} - 1) + 9 \\ &= 7 \cdot 9q + 3(7-1) \sum_{j=0}^k 7^j + 9 = 9 \left(7q + 2 \sum_{j=0}^k 7^j + 1 \right) \end{aligned}$$

Since $q \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have $7q + 2 \sum_{j=0}^k 7^j + 1 \in \mathbb{Z}$. Therefore $7^{k+1}[3(k+1)+1] - 1$ is divisible by 9.

Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

2. Solution.

Suppose $\{a_n\}_{n=0}^{\infty}$ is an infinite sequence of complex numbers. Apply mathematical induction to prove the statements below:

(a) Denote by $P(n)$ the proposition

$$\sum_{k=0}^n (a_{k+1} - a_k) = a_{n+1} - a_0.$$

- We have $\sum_{k=0}^0 (a_{k+1} - a_k) = a_1 - a_0 = a_{0+1} - a_0$.

Therefore $P(0)$ is true.

- Let $m \in \mathbb{N}$. Suppose $P(m)$ is true. Then $\sum_{k=0}^m (a_{k+1} - a_k) = a_{m+1} - a_0$.

We verify $P(m+1)$:

We have

$$\begin{aligned} \sum_{k=0}^{m+1} (a_{k+1} - a_k) &= \sum_{k=0}^m (a_{k+1} - a_k) + (a_{m+2} - a_{m+1}) \\ &= (a_{m+1} - a_0) + (a_{m+2} - a_{m+1}) = a_{m+2} - a_0 = a_{(m+1)+1} - a_0. \end{aligned}$$

Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

(b) Further suppose $a_j \neq 0$ for each $j \in \mathbb{N}$.

Denote by $P(n)$ the proposition

$$\prod_{k=0}^n (a_{k+1} - a_k) = \frac{a_{n+1}}{a_0}.$$

- We have $\prod_{k=0}^0 (a_{k+1} - a_k) = \frac{a_1}{a_0} = \frac{a_{0+1}}{a_0}$.

Therefore $P(0)$ is true.

- Let $m \in \mathbb{N}$. Suppose $P(m)$ is true. Then $\prod_{k=0}^m (a_{k+1} - a_k) = \frac{a_{m+1}}{a_0}$.

We verify $P(m+1)$:

We have

$$\prod_{k=0}^{m+1} \frac{a_{k+1}}{a_k} = \left(\prod_{k=0}^m \frac{a_{k+1}}{a_k} \right) \cdot \frac{a_{m+2}}{a_{m+1}} = \frac{a_{m+1}}{a_0} \cdot \frac{a_{m+2}}{a_{m+1}} = \frac{a_{m+2}}{a_0} = \frac{a_{(m+1)+1}}{a_0}.$$

Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

3. Solution.

(a) Denote by $P(n)$ the proposition $\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \sum_{k=n+1}^{2n} \frac{1}{k}$.

• We have $\sum_{k=1}^{2 \cdot 1} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} = \frac{1}{2} = \sum_{k=1+1}^{2 \cdot 1} \frac{1}{k}$.

Hence $P(1)$ is true.

• Let m be a positive integer. Suppose $P(m)$ is true. We deduce that $P(m+1)$ is true:

$$\begin{aligned} & \sum_{k=1}^{2(m+1)} \frac{(-1)^{k+1}}{k} - \sum_{k=(m+1)+1}^{2(m+1)} \frac{1}{k} \\ &= \sum_{k=1}^{2m} \frac{(-1)^{k+1}}{k} + \frac{(-1)^{2m+1+1}}{2m+1} + \frac{(-1)^{2m+2+1}}{2m+2} - \sum_{k=m+1}^{2m} \frac{1}{k} + \frac{1}{m+1} - \frac{1}{2m+1} - \frac{1}{2m+2} \\ &= \dots = 0 \end{aligned}$$

$$\text{Then } \sum_{k=1}^{2m} \frac{(-1)^{k+1}}{k} = \sum_{k=m+1}^{2m} \frac{1}{k}.$$

Hence $P(m+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any positive integer n .

(b) i. Let x be a real number. Suppose $x > 1$.

$$\text{We have } \frac{1}{x} = \int_{x-1}^x \frac{dt}{x} < \int_{x-1}^x \frac{dt}{t} = \ln(x) - \ln(x-1) = \ln\left(\frac{x}{x-1}\right).$$

$$\text{We also have } \ln\left(\frac{x+1}{x}\right) = \ln(x+1) - \ln(x) = \int_x^{x+1} \frac{dt}{t} < \int_x^{x+1} \frac{dt}{x} = \frac{1}{x}$$

$$\text{Therefore } \ln\left(\frac{x+1}{x}\right) < \frac{1}{x} < \ln\left(\frac{x}{x-1}\right).$$

ii. Let n be a positive integer.

$$\text{For each } k = n+1, n+2, \dots, 2n, \text{ we have } \ln\left(\frac{k+1}{k}\right) < \frac{1}{k} < \ln\left(\frac{k}{k-1}\right).$$

$$\text{Then } \sum_{k=n+1}^{2n} \frac{1}{k} > \sum_{k=n+1}^{2n} \ln\left(\frac{k+1}{k}\right) = \ln\left(\prod_{k=n+1}^{2n} \frac{k+1}{k}\right) = \ln\left(\frac{2n+1}{n+1}\right).$$

$$\text{Also } \sum_{k=n+1}^{2n} \frac{1}{k} < \sum_{k=n+1}^{2n} \ln\left(\frac{k}{k-1}\right) = \ln\left(\prod_{k=n+1}^{2n} \frac{k}{k-1}\right) = \ln(2)$$

$$\text{Therefore } \ln\left(\frac{2n+1}{n+1}\right) < \sum_{k=n+1}^{2n} \frac{1}{k} < \ln(2).$$

iii. Let n be a positive integer. By the results in the previous parts, we have $\ln\left(2 - \frac{1}{n+1}\right) = \ln\left(\frac{2n+1}{n+1}\right) <$

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} < \ln(2).$$

By the continuity of the logarithmic function, $\lim_{n \rightarrow \infty} \ln\left(2 - \frac{1}{n+1}\right) = \ln(2)$. Also $\lim_{n \rightarrow \infty} \ln(2) = \ln(2)$.

By the Sandwich Rule, the limit $\lim_{n \rightarrow \infty} \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k}$ exists and is equal to $\ln(2)$.

4. Answer.

(I) Suppose $\sum_{j=0}^n a_j = \left(\frac{1+a_n}{2}\right)^2$ for each $n \in \mathbb{N}$.

(II) $a_n = 2n + 1$.

(III) We have $a_0 = \sum_{j=0}^0 a_j = \left(\frac{1+a_0}{2}\right)^2 = \frac{1}{4}(1+2a_0+a_0^2)$. Then $(a_0 - 1)^2 = a_0^2 - 2a_0 + 1 = 0$. Therefore $a_0 = 1 = 2 \cdot 0 + 1$.

(IV) Let $k \in \mathbb{N}$. Suppose $P(k)$ is true.

(V) We have

$$\left(\frac{1+a_{k+1}}{2}\right)^2 = \sum_{j=0}^{k+1} a_j = \sum_{j=0}^k a_j + a_{k+1} = \left(\frac{1+a_k}{2}\right)^2 + a_{k+1} = \left[\frac{1+(2k+1)}{2}\right]^2 + a_{k+1} = (k+1)^2 + a_{k+1}.$$

Then $\frac{1}{4}(1+2a_{k+1}+a_{k+1}^2) = (k+1)^2 + a_{k+1}$.

Therefore $(a_{k+1}-1)^2 = a_{k+1}^2 - 2a_{k+1} + 1 = (2k+2)^2$.

Hence $a_{k+1} = 2k+3$ or $a_{k+1} = -2k-1$. Since $a_{k+1} > 0$, we have $a_{k+1} = 2k+3 = 2(k+1)+1$.

(VI) By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

5. Answer.

(I) $1 = -(-1) = \alpha + \beta$

(II) $3 = [-(-1)]^2 - 2(-1) = (\alpha + \beta)^2 - 2\alpha\beta = \alpha^2 + \beta^2$

(III) Let k be a positive integer. Suppose $P(k)$ is true.

(IV) $\alpha^{k+1} + \beta^{k+1}$

(V) $P(k)$

(VI) $a_{k+2} = a_{k+1} + a_k = (\alpha^{k+1} + \beta^{k+1}) + (\alpha^k + \beta^k) = \alpha^k(\alpha + 1) + \beta^k(\beta + 1) = \alpha^k \cdot \alpha^2 + \beta^k \cdot \beta^2 = \alpha^{k+2} + \beta^{k+2}$.

(VII) By the Principle of Mathematical Induction, $P(n)$ is true for each positive integer n .

6. Solution.

Suppose $a \in (-1, +\infty)$. Denote by $P(n)$ the proposition $(1+a)^n \geq 1+na$.

- We have $(1+a)^2 = 1+2a+a^2 \geq 1+2 \cdot a$.
Hence $P(2)$ is true.
- Let $k \in \mathbb{N} \setminus \{0, 1\}$. Suppose $P(k)$ is true. Then $(1+a)^k \geq 1+ka$.
We verify that $P(k+1)$ is true:

$$\begin{aligned} (1+a)^{k+1} &= (1+a)^k(1+a) \\ &\geq (1+ka)(1+a) \quad \text{because } 1+a \geq 0, \\ &= 1+(k+1)a+ka^2 \\ &\geq 1+(k+1)a \quad \text{because } ka^2 \geq 0. \end{aligned}$$

Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N} \setminus \{0, 1\}$.

7. Solution.

(a) (I) Suppose $\mu_1, \dots, \mu_n \in \mathbb{C}$. Then $\left| \sum_{j=1}^n \mu_j \right| \leq \sum_{j=1}^n |\mu_j|$.

(II) Let μ_1, μ_2 be complex numbers.

$$\left| \sum_{j=1}^2 \mu_j \right| = |\mu_1 + \mu_2| \leq |\mu_1| + |\mu_2| = \sum_{j=1}^2 |\mu_j|.$$

(III) Let $k \in \mathbb{N} \setminus \{0, 1\}$. Suppose $P(k)$ is true.

(IV) Let $\nu_1, \dots, \nu_k, \nu_{k+1}$ be complex numbers. We have

$$\left| \sum_{j=1}^{k+1} \nu_j \right| = \left| \sum_{j=1}^k \nu_j + \nu_{k+1} \right| \leq \left| \sum_{j=1}^k \nu_j \right| + |\nu_{k+1}| \leq \sum_{j=1}^k |\nu_j| + |\nu_{k+1}| \leq \sum_{j=1}^{k+1} |\nu_j|$$

(b) Let $\zeta \in \mathbb{C}$. Suppose $0 < |\zeta| < 1$. Then we have

$$\left| \sum_{k=1050}^{4060} \zeta^k \right| \leq \sum_{k=1050}^{4060} |\zeta^k| = \sum_{k=1050}^{4060} |\zeta|^k = |\zeta|^{1050} \cdot \sum_{k=0}^{3010} |\zeta|^k = |\zeta|^{1050} \cdot \frac{1-|\zeta|^{3011}}{1-|\zeta|} < \frac{|\zeta|^{1050}}{1-|\zeta|}.$$

The first inequality is a consequence of Statement (T). The last inequality follows from $|\zeta|^{1050} > 0$ and $0 < |\zeta|^{3011} < 1$.

8. (a) *Hint.* The appropriate proposition $P(n)$ upon which mathematical induction is applied is:
 Suppose b_1, b_2, \dots, b_n are positive real numbers. Then $(1 + b_1)(1 + b_2) \cdots (1 + b_n) > 1 + (b_1 + b_2 + \cdots + b_n)$.
- (b) *Hint.* Be aware that whenever $0 < p < 1$, the inequalities $0 < 1 - p < 1$ and $0 < (1 - p)(1 + p) < 1$ hold.

9. **Solution.**

- (a) Let a, b, u, v be positive real numbers. Suppose $u + v = 1$.

We have

$$\begin{aligned} a^2u + b^2v - (au + bv)^2 &= a^2u + b^2v - a^2u^2 - b^2v^2 - 2abuv \\ &= a^2u(1 - u) + b^2v(1 - v) - 2abuv = a^2uv + b^2uv - 2abuv = (a - b)^2uv \geq 0. \end{aligned}$$

Then $a^2u + b^2v \geq (au + bv)^2$. Therefore $\sqrt{a^2u + b^2v} \geq au + bv$.

- (b) Denote by $P(n)$ the statement below:

Suppose $c_1, c_2, \dots, c_n, x_1, x_2, \dots, x_n$ be positive real numbers. Further suppose $x_1 + x_2 + \cdots + x_n = 1$. Then

$$\sqrt{c_1^2x_1 + c_2^2x_2 + \cdots + c_n^2x_n} \geq c_1x_1 + c_2x_2 + \cdots + c_nx_n.$$

- By the result in part (a), $P(2)$ is true.
- Let $k \in \mathbb{N} \setminus \{0, 1\}$. Suppose $P(k)$ is true.

We verify that $P(k + 1)$ is true:

Suppose $c_1, c_2, \dots, c_k, c_{k+1}, x_1, x_2, \dots, x_k, x_{k+1}$ be positive real numbers. Further suppose $x_1 + x_2 + \cdots + x_k + x_{k+1} = 1$.

Define $t = x_1 + x_2 + \cdots + x_k$. We have $t > 0$ and $t + x_{k+1} = 1$.

For each $j = 1, 2, \dots, k$, define $u_j = \frac{x_j}{t}$. Note that u_1, u_2, \dots, u_k are positive real numbers and

$$u_1 + u_2 + \cdots + u_k = 1.$$

Define $d = \sqrt{c_1^2u_1 + c_2^2u_2 + \cdots + c_k^2u_k}$. By definition, $d^2t = c_1^2x_1 + c_2^2x_2 + \cdots + c_k^2x_k$

By $P(k)$, we have $d \geq c_1u_1 + c_2u_2 + \cdots + c_ku_k$.

Now

$$\begin{aligned} \sqrt{c_1^2x_1 + c_2^2x_2 + \cdots + c_k^2x_k + c_{k+1}^2x_{k+1}} &= \sqrt{d^2t + c_{k+1}^2x_{k+1}} \\ &\geq dt + c_{k+1}x_{k+1} \quad (\text{by } P(2)) \\ &= (c_1u_1 + c_2u_2 + \cdots + c_ku_k)t + c_{k+1}x_{k+1} \\ &= c_1x_1 + c_2x_2 + \cdots + c_kx_k + c_{k+1}x_{k+1} \end{aligned}$$

Hence $P(k + 1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N} \setminus \{0, 1\}$.

10. (a) **Answer.**

- (I) x is an irrational number
- (II) it were true that \sqrt{x} was a rational number
- (III) x is positive
- (IV) \sqrt{x} was a rational number
- (V) a rational number
- (VI) an irrational number
- (VII) assumption
- (VIII) false

(b) **Solution.**

- i. Let x be a positive real number, r be a positive rational number, and n be an integer greater than 1. Suppose x is an irrational number.

Further suppose it were true that $\sqrt[n]{x + r}$ was a rational number.

Write $y = \sqrt[n]{x + r}$. Note that $y^n - r = x$.

Since y was a rational number, y^n would be a rational number. Moreover, since r is a rational number, $y^n - r$ would be a rational number.

Therefore x would be a rational number. But by assumption x is an irrational number. Contradiction arises.

Hence the assumption that $\sqrt[n]{x + r}$ was rational is false. $\sqrt[n]{x + r}$ is an irrational number in the first place.

- ii. Let $r, s, t \in \mathbb{R}$. Suppose r is a non-zero rational number and s is an irrational number.

Further suppose it were true that both $rs + t, rs - t$ were rational numbers.

Note that $2rs = (rs + t) + (rs - t)$. Then $2rs$ would be a rational number.

Since $2, r$ are non-zero rational numbers, $2r$ is a non-zero rational number.

Note that $s = \frac{2rs}{2r}$. Then s would be a rational number.

But s is an irrational number.

Contradiction arises.

Hence at least one of $rs + t, rs - t$ is irrational.

11. (a) **Answer.**

- (I) Suppose it were true that $\sqrt[3]{3}$ was not irrational
- (II) there would exist some $m, n \in \mathbb{Z}$
- (III) $n \neq 0$ and $m = n \cdot \sqrt[3]{3}$
- (IV) m^3 would be divisible by 3
- (V) Euclid's Lemma
- (VI) there would exist some $k \in \mathbb{Z}$ such that $m = 3k$
- (VII) Note that $3k^3$ was an integer. Then n^3 would be divisible by 3.
- (VIII) 3 is a prime number
- (IX) n would be divisible by 3
- (X) m, n have no common factors other than $-1, 1$

(b) i. **Solution.**

Suppose it were true that $\sqrt[5]{7}$ was not irrational.

Then $\sqrt[5]{7}$ would be a rational number. There would exist some $m, n \in \mathbb{Z}$ such that $n \neq 0$ and $\sqrt[5]{7} = \frac{m}{n}$.

Without loss of generality, we may assume that m, n have no common factors other than 1, -1 .

We would have $m = \sqrt[5]{7}n$. Then $m^5 = 7n^5$.

Now m^5 was divisible by 7. Also note that 7 is a prime number. By Euclid's Lemma, m would be divisible by 7. Therefore there existed some $k \in \mathbb{Z}$ such that $m = 7k$.

Then we would have $7^5k^5 = (7k)^5 = m^5 = 7n^5$. Therefore $n^5 = 7^4k^5 = 7(7^3k^5)$.

Now n^5 was divisible by 7. Also note that 7 is a prime number. By Euclid's Lemma, n would be divisible by 7.

Therefore both m, n would be divisible by 7. 7 would be a common factor of m, n . Recall that we assumed that m, n have no common factors other than $-1, 1$.

Contradiction arises.

Therefore the assumption that $\sqrt[5]{7}$ was not irrational is false. $\sqrt[5]{7}$ is irrational.

ii. —

12. —