MATH1050 Exercise 4

1. Apply mathematical induction to justify each of the statements below:

   (a) $1 \cdot 2 + 2 \cdot 5 + 3 \cdot 8 + \cdots + n(3n-1) = n^2(n+1)$ *for any positive integer* n.

   (b) $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}} \geq \sqrt{n}$ *whenever* n *is a positive integer.*

   (c) $n^2 < 2^{n-1}$ *whenever* n *is an integer greater than* 6.

   (d) $n(n^2+2)$ *is divisible by* 3 *for any* $n \in \mathbb{N}$.

   (e) $7^n(3n+1) - 1$ *is divisible by* 9 *for any* $n \in \mathbb{N}$.

2. Suppose $\{a_n\}_{n=0}^{\infty}$ is an infinite sequence of complex numbers. Apply mathematical induction to prove the statements below:

   (a) $\displaystyle\sum_{k=0}^{n}(a_{k+1} - a_k) = a_{n+1} - a_0$.

   (b) *Further suppose* $a_j \neq 0$ *for each* $j \in \mathbb{N}$. *Then* $\displaystyle\prod_{k=0}^{n} \frac{a_{k+1}}{a_k} = \frac{a_{n+1}}{a_0}$.

   Remarks. The results proved here give the mechanism for a useful method for computing sums/products of consecutive terms of sequences. This method is known as the **Telescopic Method**.

3. (a)$^\diamondsuit$ Apply mathematical induction to prove that $\displaystyle\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \sum_{k=n+1}^{2n} \frac{1}{k}$ for any positive integer n.

   (b)$^\clubsuit$ *In this part you are assumed to be familiar with calculus of one real variable.*

   i. *Take for granted the validity of the result below about definite integrals (which looks 'obvious' in terms of the ' area interpretation' for definite integration):*

   - *Let* a, b *be real numbers, with* $a < b$, *and let* f, g *be real-valued functions of one real variable whose domains contain the interval* $[a, b]$. *Suppose* f, g *are continuous on* $[a, b]$. *Further suppose that* $f(x) \leq g(x)$ *for any* $x \in [a, b]$, *and also suppose that there exists some* $x_0 \in [a, b]$ *such that* $f(x_0) < g(x_0)$. *Then* $\displaystyle\int_a^b f(t)dt < \int_a^b g(t)dt$.

   Prove the statement ($\sharp$):

   ($\sharp$) *Let* x *be a real number. Suppose* $x > 1$. *Then* $\ln\left(\dfrac{x+1}{x}\right) < \dfrac{1}{x} < \ln\left(\dfrac{x}{x-1}\right)$.

   ii. Applying the statement ($\sharp$), or otherwise, deduce that $\ln\left(\dfrac{2n+1}{n+1}\right) < \displaystyle\sum_{k=n+1}^{2n} \frac{1}{k} < \ln(2)$ for any positive integer n.

   iii. Hence, or otherwise, prove that the limit $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k}$ exists and find its value.

   Remark. We can further deduce $\displaystyle\lim_{m\to\infty} \sum_{k=1}^{m} \frac{(-1)^{k+1}}{k}$ exists and is equal to the limit $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k}$.

4. Consider the statement (S):

   (S) *Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of positive real numbers. Suppose* $\displaystyle\sum_{j=0}^{n} a_j = \left(\frac{1+a_n}{2}\right)^2$ *for each $n \in \mathbb{N}$. Then $a_n = 2n+1$ for each $n \in \mathbb{N}$.*

   Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate passages so that it gives an argument by mathematical induction for the statement (S).

   > Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of positive real numbers. ______(I)______
   >
   > Denote by $P(n)$ the proposition below:
   >
   > ______(II)______
   >
   > - We verify that $P(0)$ is true:
   >
   >   ______(III)______
   >
   >   Hence $P(0)$ is true.
   > - ______(IV)______
   >
   >   We verify that $P(k+1)$ is true:
   >
   >   ______(V)______
   >
   >   Therefore $P(k+1)$ is true.
   >
   > ______(VI)______

5. Consider the statement (Q):

   - *Let α, β are the two distinct roots of the polynomial $f(x) = x^2 - x - 1$. Suppose $\{a_n\}_{n=1}^{\infty}$ is the infinite sequence of real numbers defined by*

   $$\begin{cases} a_1 = 1, & a_2 = 3, \\ & a_{n+2} = a_{n+1} + a_n \quad \text{if} \quad n \geq 1 \end{cases}.$$

   Then $a_n = \alpha^n + \beta^n$ for each positive integer n.

   Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate passages so that it gives an argument by mathematical induction for the statement (Q).

   > Let α, β are the two distinct roots of the polynomial $f(x) = x^2 - x - 1$. Suppose $\{a_n\}_{n=1}^{\infty}$ is the infinite sequence of real numbers defined by
   >
   > $$\begin{cases} a_1 = 1, & a_2 = 3, \\ & a_{n+2} = a_{n+1} + a_n \quad \text{if} \quad n \geq 1 \end{cases}.$$
   >
   > Denote by $P(n)$ the proposition below:
   >
   > $a_n = \alpha^n + \beta^n$ and $a_{n+1} = \alpha^{n+1} + \beta^{n+1}$.
   >
   > - We verify that $P(1)$ is true:
   >
   >   We have $a_1 =$ ______(I)______ .
   >
   >   We also have $a_2 =$ ______(II)______ .
   >
   >   Hence $P(1)$ is true.
   > - ______(III)______
   >
   >   Then $a_k = \alpha^k + \beta^k$, and $a_{k+1} = \alpha^{k+1} + \beta^{k+1}$.
   >
   >   We verify that $P(k+1)$ is true:
   >
   >   We have $a_{k+1} =$ ______(IV)______ by ___(V)___ immediately.
   >
   >   Now we verify that $a_{(k+1)+1} = \alpha^{(k+1)+1} + \beta^{(k+1)+1}$:
   >
   >   ______(VI)______
   >
   >   Therefore $P(k+1)$ is true.
   >
   > ______(VII)______

6. Apply mathematical induction to prove **Bernoulli's Inequality** in the formulation below:

   - *Suppose $a \in (-1, +\infty)$. Then $(1+a)^n \geq 1 + na$ for any $n \in \mathbb{N} \backslash \{0, 1\}$.*

7. (a) *Here you may tacitly assume the result that $|\mu + \nu| \leq |\mu| + |\nu|$ for any $\mu, \nu \in \mathbb{C}$. For the proof of this result, refer to Assignment 3.*
   Consider the statement (T):

   (T) *Let $n \in \mathbb{N} \backslash \{0, 1\}$. Suppose $\mu_1, \mu_2, \cdots, \mu_n \in \mathbb{C}$. Then* $\left| \sum_{j=1}^{n} \mu_j \right| \leq \sum_{j=1}^{n} |\mu_j|$.

   Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate passages so that it gives an argument by mathematical induction for the statement (T).

   > Denote by $P(n)$ the proposition below:
   >
   > (I) ____________________
   >
   > - We verify that $P(2)$ is true:
   >
   >   (II) ____________________
   >
   >   Hence $P(2)$ is true.
   > - (III) ____________________
   >
   >   We verify that $P(k+1)$ is true:
   >
   >   (IV) ____________________
   >
   >   Therefore $P(k+1)$ is true.
   >
   > By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N} \backslash \{0, 1\}$.

   Remark. The statement (T) is the **General Triangle Inequality for complex numbers**.

   (b) Let $\zeta \in \mathbb{C}$. Suppose $0 < |\zeta| < 1$. By applying the results above, or otherwise, prove the inequality

   $$\left| \sum_{k=1050}^{4060} \zeta^k \right| < \frac{|\zeta|^{1050}}{1 - |\zeta|}.$$

8. (a)$^\diamondsuit$ Apply mathematical induction to prove the statement $(\sharp)$:

   $(\sharp)$ *Let $n \in \mathbb{N} \backslash \{0, 1\}$. Suppose $b_1, b_2, \cdots, b_n$ are positive real numbers. Then $(1+b_1)(1+b_2) \cdot \ldots \cdot (1+b_n) > 1 + (b_1 + b_2 + \cdots + b_n)$.*

   (b) By applying the result above, or otherwise, prove the statement $(\flat)$:

   $(\flat)$ *Let $n \in \mathbb{N} \backslash \{0, 1\}$. Suppose $b_1, b_2, \cdots, b_n \in (0, 1)$. Then* $(1-b_1)(1-b_2) \cdot \ldots \cdot (1-b_n) < \dfrac{1}{1 + (b_1 + b_2 + \cdots + b_n)}$.

   Remark. These are two of **Weierstrass's Product Inequalities**.

9. (a) Let a, b, u, v be positive real numbers. Suppose $u + v = 1$.
   Prove that $\sqrt{a^2 u + b^2 v} \geq au + bv$.

   (b)$^\clubsuit$ Prove the statement below:

   - *Let $n \in \mathbb{N} \backslash \{0, 1\}$. Suppose $c_1, c_2, \cdots, c_n, x_1, x_2, \cdots, x_n$ be positive real numbers. Further suppose $x_1 + x_2 + \cdots + x_n = 1$. Then*
     $$\sqrt{{c_1}^2 x_1 + {c_2}^2 x_2 + \cdots + {c_n}^2 x_n} \geq c_1 x_1 + c_2 x_2 + \cdots + c_n x_n.$$

10. *In this question, you may tacitly assumed the results that the sum and the product of any pairs of rational numbers are rational numbers, the difference of one rational number from another is a rational number, and the quotient of one rational number by a non-zero rational number is also a rational number. For the proofs of these results, refer to Assignment 1.*

    (a) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate passages so that it gives a proof-by-contradiction argument for the statement (I).

    (I) *Let x be a positive real number. Suppose x is an irrational number. Then $\sqrt{x}$ is an irrational number.*

> Let x be a positive real number.
> Suppose ________(I)________ .
> Further suppose ________(II)________ .
> Since ____(III)____ , we have $x = (\sqrt{x})^2$.
> Since ____(IV)____ , $(\sqrt{x})^2$ would be a rational number as well.
> Therefore x would be ____(V)____ .
> But x is assumed to be ____(VI)____ .
> Contradiction arises.
> Hence the ___(VII)___ that $\sqrt{x}$ was rational is ___(VIII)___ . It follows that $\sqrt{x}$ is an irrational number in the first place.

(b) Apply proof-by-contradiction to justify each of the statements below.

i. *Let x be a positive real number, r be a positive rational number, and n be an integer greater than 1. Suppose x is an irrational number. Then $\sqrt[n]{x+r}$ is an irrational number.*

ii.$^\diamondsuit$ *Let $r, s, t \in \mathbb{R}$. Suppose r is a non-zero rational number and s is an irrational number. Then at least one of $rs + t$, $rs - t$ is an irrational number.*

11. *In this question you may take for granted the validity of Euclid's Lemma:*

- *Let $h, k, p \in \mathbb{Z}$. Suppose p is a prime number. Further suppose hk is divisible by p. Then at least one of h, k is divisible by p.*

(a) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate passages so that it gives a proof-by-contradiction argument for the statement (J).

(J) *$\sqrt[3]{3}$ is irrational.*

> ________(I)________ .
> Then $\sqrt[3]{3}$ would be a rational number.
> Therefore ______(II)______ such that ______(III)______ .
> Without loss of generality, we may assume that m, n have no common factors other than $1, -1$.
> Since $m = n \cdot \sqrt[3]{3}$, we would have $m^3 = 3n^3$.
> Note that n^3 was an integer. Then ________(IV)________ .
> Now also note that 3 is a prime number. Then, by ____(V)____ , m would be divisible by 3.
> Therefore ________(VI)________ .
> Then we would have $27k^3 = (3k)^3 = m^3 = 3n^3$. Therefore $n^3 = 9k^3 = 3(3k^3)$.
> ________(VII)________
> Note that ______(VIII)______ . Then, by Euclid's Lemma, ______(IX)______ .
> Therefore both m, n would be divisible by 3. Hence 3 would be a common factor of m, n.
> Recall that we have assumed that ______(X)______ .
> Contradiction arises.
> Therefore the assumption that $\sqrt[3]{3}$ was not irrational is false. It follows that $\sqrt[3]{3}$ is irrational in the first place.

(b) Apply proof-by-contradiction to justify each of the statements below.

i. *$\sqrt[5]{7}$ is irrational.*

ii.$^\diamondsuit$ *Let p be a positive prime number, and Q be an integer greater than 1. The number $\sqrt[Q]{p}$ is irrational.*

12. Apply proof-by-contradiction to justify each of the statements below.

(a) *Let a, b be real numbers. Suppose $a > b > 0$. Then $\sqrt{a^2 - b^2} + \sqrt{2ab - b^2} > a$.*

(b) *Let a, b be real numbers. Suppose $|a| \leq 1$ and $|b| \leq 1$. Then $\sqrt{1 - a^2} + \sqrt{1 - b^2} \leq 2\sqrt{1 - (a+b)^2/4}$.*

(c)$^\diamondsuit$ *Let a, b be real numbers. Suppose $ab \neq 0$. Then $\left|\dfrac{a + \sqrt{a^2 + 2b^2}}{2b}\right| < 1$ or $\left|\dfrac{a - \sqrt{a^2 + 2b^2}}{2b}\right| < 1$.*

(d)$^\diamondsuit$ *Let a, b be real numbers. Suppose $f(x)$ is the quadratic polynomial given by $f(x) = x^2 + ax + b$. Then $|f(1)| \geq \dfrac{1}{2}$ or $|f(2)| \geq \dfrac{1}{2}$ or $|f(3)| \geq \dfrac{1}{2}$.*

(*Hint.* Can you find a relation amongst $f(1), f(2), f(3)$ which does not involve a, b?)