MATH1050 Exercise 3 (Answers and solution)

1. **Solution.**
   (a) Let ζ be a complex numbers. We have $\zeta\bar{\zeta} = (\mathsf{Re}(\zeta) + i\mathsf{Im}(\zeta))(\mathsf{Re}(\zeta) - i\mathsf{Im}(\zeta)) = (\mathsf{Re}(\zeta))^2 + (\mathsf{Im}(\zeta))^2 = |\zeta|^2$.
   (b) Let z, w be complex numbers. Suppose $w \neq 0$. $\dfrac{z}{w} = \dfrac{z\bar{w}}{w\bar{w}} = \dfrac{z\bar{w}}{|w|^2}$.

2. **Solution.**
   Let z, w be complex numbers.
   (a) $|zw|^2 = (zw)\overline{(zw)} = zw\bar{z}\bar{w} = (z\bar{z})(w\bar{w}) = |z|^2|w|^2$.
   Since $|z| \geq 0$, $|w| \geq 0$ and $|zw| \geq 0$, we have $|zw| = |z||w|$.
   (b) Suppose $z \neq 0$ and $w \neq 0$, and θ, φ are respective arguments of z, w.
   Then $z = |z|(\cos(\theta) + i\sin(\theta))$ and $w = |w|(\cos(\varphi) + i\sin(\varphi))$.
   Therefore
$$\begin{aligned} zw &= [|z|(\cos(\theta) + i\sin(\theta))][|w|(\cos(\varphi) + i\sin(\varphi))] \\ &= |z||w|[(\cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)) + i(\sin(\theta)\cos(\varphi) + \cos(\theta)\sin(\varphi))] \\ &= |zw|(\cos(\theta+\varphi) + i\sin(\theta+\varphi)). \end{aligned}$$

3. **Solution.**
   Let $\omega = \dfrac{\sqrt{3}+i}{2}$.
   (a) $\omega^2 = \dfrac{1+\sqrt{3}i}{2}$, $\omega^3 = i$, $\omega^{11} = \dfrac{\sqrt{3}-i}{2}$, $\omega^{12} = 1$.
   (b) $\displaystyle\sum_{k=0}^{2230} \omega^{k+1} = \omega \sum_{k=0}^{2230} \omega^k = \omega \cdot \frac{1-\omega^{2231}}{1-\omega} = \omega \cdot \frac{1-\omega^{-1}}{1-\omega} = -1$.

4. **Solution.**
   Let a, b, c be real numbers. Suppose $a^2+b^2+c^2 = 1$ and $c \neq 1$. Define $z = \dfrac{a+bi}{1-c}$.
   (a) $|z|^2 = z\bar{z} = \dfrac{a+bi}{1-c} \cdot \dfrac{a-bi}{1-c} = \dfrac{a^2+b^2}{(1-c)^2} = \dfrac{1-c^2}{(1-c)^2} = \dfrac{1+c}{1-c}$.
   (b) We have $|z|^2 = \dfrac{1+c}{1-c}$.
   Then $c = \dfrac{|z|^2-1}{|z|^2+1} = \dfrac{z\bar{z}-1}{z\bar{z}+1}$.
   Since $z = \dfrac{a+bi}{1-c}$, we have $\mathsf{Re}(z) = \dfrac{a}{1-c}$ and $\mathsf{Im}(z) = \dfrac{b}{1-c}$.
   Therefore $a = (1-c)\mathsf{Re}(z) = \left(1 - \dfrac{z\bar{z}-1}{z\bar{z}+1}\right) \cdot \dfrac{z+\bar{z}}{2} = \dfrac{z+\bar{z}}{z\bar{z}+1}$.
   Also, $b = (1-c)\mathsf{Im}(z) = \left(1 - \dfrac{z\bar{z}-1}{z\bar{z}+1}\right) \cdot \dfrac{z-\bar{z}}{2i} = \dfrac{z-\bar{z}}{i(z\bar{z}+1)}$.

5. **Solution.**
   (a) Suppose z, w are complex numbers.
   Then $|z+w|^2 = |z|^2 + |w|^2 + z\bar{w} + \bar{z}w$.
   Also, $|z-w|^2 = |z+(-w)|^2 = |z|^2 + |-w|^2 + z\overline{(-w)} + \bar{z}(-w) = |z|^2 + |w|^2 - z\bar{w} - \bar{z}w$.
   Then $|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2 + z\bar{w} + \bar{z}w - z\bar{w} - \bar{z}w = 2|z|^2 + 2|w|^2$.
   (b) i. Suppose r, s, t are complex numbers.
   Then $|2r-s-t|^2 = |(r-s)+(r-t)|^2 = 2|r-s|^2 + 2|r-t|^2 - |(r-s)-(r-t)|^2 = 2|r-s|^2 + 2|t-r|^2 - |s-t|^2$.
   Similarly, $|2s-t-r|^2 = 2|s-t|^2 + 2|r-s|^2 - |t-r|^2$ and $|2t-r-s|^2 = 2|t-r|^2 + 2|s-t|^2 - |r-s|^2$.
   Therefore $|2r-s-t|^2 + |2s-t-r|^2 + |2t-r-s|^2 = 3(|s-t|^2 + |t-r|^2 + |r-s|^2)$.
   ii. Let ζ, α, β be complex numbers. Suppose $\zeta^2 = \alpha^2 + \beta^2$. Then $(\zeta+\alpha)(\zeta-\alpha) = \zeta^2 - \alpha^2 = \beta^2$.
   We have
$$\begin{aligned} (|\zeta+\alpha| + |\zeta-\alpha|)^2 &= |\zeta+\alpha|^2 + |\zeta-\alpha|^2 + 2|\zeta+\alpha||\zeta-\alpha| \\ &= 2|\zeta|^2 + 2|\alpha|^2 + 2|(\zeta+\alpha)(\zeta-\alpha)| \\ &= 2|\zeta|^2 + 2|\alpha|^2 + 2|\beta^2| \\ &= 2|\zeta|^2 + 2|\alpha|^2 + 2|\beta|^2 \end{aligned}$$

Modifying the above argument (by interchanging the roles played by α and β), we have $(|\zeta+\beta|+|\zeta-\beta|)^2 = 2|\zeta|^2+2|\beta|^2+2|\alpha|^2$.
Therefore $(|\zeta+\alpha|+|\zeta-\alpha|)^2 = (|\zeta+\beta|+|\zeta-\beta|)^2$.
Note that $|\zeta+\alpha|+|\zeta-\alpha| \geq 0$ and $|\zeta+\beta|+|\zeta-\beta| \geq 0$. Then $|\zeta+\alpha|+|\zeta-\alpha| = |\zeta+\beta|+|\zeta-\beta|$.

6. **Solution.**

Let $z \in \mathbb{C}$. Suppose $4|z+1| = |z+16|$. Then $16(z+1)(\bar{z}+1) = 16(z+1)\overline{(z+1)} = 16|z+1|^2 = |z+16|^2 = (z+16)\overline{(z+16)} = (z+16)(\bar{z}+16)$.

Now $16(z\bar{z}+z+\bar{z}+1) = z\bar{z}+16z+16\bar{z}+16^2$.

Therefore $15z\bar{z} = 15\cdot 16$. We have $|z|^2 = z\bar{z} = 16$. Hence $|z| = 4$. z lies on the circle with centre at 0 and radius 4.

7. **Solution.**
Write $\mu = 1+i$, $\nu = 3-i$.

(a) Note that $\dfrac{\mathsf{Im}(\nu)-\mathsf{Im}(\mu)}{\mathsf{Re}(\nu)-\mathsf{Re}(\mu)} = \dfrac{-1-1}{3-1} = -1$.
Let $\eta \in \mathbb{C}$.

$$\begin{aligned}
&\eta \text{ lies on the ('infinite') straight line joining } \mu, \nu \\
\text{iff}\quad & \mathsf{Im}(\eta)-\mathsf{Im}(\mu) = \frac{\mathsf{Im}(\nu)-\mathsf{Im}(\mu)}{\mathsf{Re}(\nu)-\mathsf{Re}(\mu)}\cdot(\mathsf{Re}(\eta)-\mathsf{Re}(\mu)) \\
\text{iff}\quad & \mathsf{Im}(\eta)-1 = (-1)\cdot(\mathsf{Re}(\eta)-1) \\
\text{iff}\quad & \mathsf{Re}(\eta)+\mathsf{Im}(\eta)-2 = 0
\end{aligned}$$

(b) Consider the curve C on the Argand plane defined by the equation $|z-\mu| = |z-\nu|$.

i. Let $\zeta \in \mathbb{C}$. Note that

$$\begin{aligned}
|\zeta-\mu|^2 &= (\zeta-\mu)\overline{(\zeta-\mu)} = \zeta\bar{\zeta}-\bar{\mu}\zeta-\mu\bar{\zeta}+\mu\bar{\mu} = |\zeta|^2-(1-i)\zeta-(1+i)\bar{\zeta}+2 \\
|\zeta-\nu|^2 &= (\zeta-\nu)\overline{(\zeta-\nu)} = \zeta\bar{\zeta}-\bar{\nu}\zeta-\nu\bar{\zeta}+\nu\bar{\nu} = \zeta\bar{\zeta}+(-3-i)\zeta+(-3+i)\bar{\zeta}+10.
\end{aligned}$$

We have

$$\begin{aligned}
|z-1-i| = |z-3+i| \quad \text{iff}\quad & |z-1-i|^2 = |z-3+i|^2 \\
\text{iff}\quad & (2+2i)\zeta+(2-2i)\bar{\zeta}-8 = 0 \\
\text{iff}\quad & 2(\zeta+\bar{\zeta})+2i(\zeta-\bar{\zeta})-8 = 0 \\
\text{iff}\quad & \frac{1}{2}(\zeta+\bar{\zeta})-\frac{1}{2i}(\zeta-\bar{\zeta})-2 = 0 \\
\text{iff}\quad & \mathsf{Re}(\zeta)-\mathsf{Im}(\zeta)-2 = 0
\end{aligned}$$

ii. The equation of the 'infinite' straight line ℓ joining μ, ν is given by $\mathsf{Re}(z)+\mathsf{Im}(z)-2 = 0$ with unknown z in the complex numbers.
The curve C is a straight line whose equation is given by $\mathsf{Re}(z)-\mathsf{Im}(z)-2 = 0$ with unknown z in the complex numbers.
The slope of ℓ is -1, and the slope of C is 1. Therefore the lines ℓ, C are perpendicular to each other.
ℓ and C intersect each other at the point 2.
Note that $|2-\mu| = \sqrt{2} = |2-\nu|$. Hence 2 is the mid-point of the line segment joining μ, ν.
It follows that C is the perpendicular bisector of the line segment joining μ, ν.

8. **Answer.**
$z = 2$ or $z = 4+2i$.

Remark. The curve described by the equation $|z-2-2i| = 2$ is the circle with centre $2+2i$ and radius 2. It is tangent to the real axis at 2 and the imaginary axis at $2i$.

The curve described by the equation $|z-4+2i| = |z-2i|$ is the 'infinite' straight line which perpendicularly bisects the line segment joining $4-2i$ and $2i$.

The latter passes the point 2, which lies on the former, and by symmetry (and perpendicularity), passes through the point $4+2i$ on the former.

9. **Answer.**

(a) $2+2i$, $4i$ (b) $\sqrt{2}$ (c) $-1+i$

Remark. The curve described by the equation $|z-2i| = 2$ is the circle with centre $2i$ and radius 2. It is tangent to the real axis at 0.
The curve described by the equation $|z-4-4i| = |z|$ is the 'infinite' straight line which perpendicularly bisects the line segment joining $4+4i$ and 0.

They intersect each other at the points $2+2i$ and $4i$ only.

Because $(S_{\alpha,r})$ has exactly two solutions, the curve $|z-\alpha| = r$, which describes the circle with centre α and with radius r, must pass through the points $2+2i$ and $4i$, no matter which values α, r take. Then by symmetry, α lies on the 'infinite' straight line which is the perpendicular bisector for the line segment joining $2+2i$ and $4i$.

10. **Solution.**

Let ω be a complex number. Suppose $|\omega| = 1$ and $\mathsf{Im}(\omega) \geq 0$. Further suppose $\omega^2 + \dfrac{5}{\omega} - 2$ is purely imaginary.

There exists some $\theta \in \mathbb{R}$ such that $\omega = |\omega|(\cos(\theta) + i\sin(\theta))$.

Since $|\omega| = 1$, we have $\omega = \cos(\theta) + i\sin(\theta)$. Then $\omega^2 = \cos(2\theta) + i\sin(2\theta)$ and $\dfrac{1}{\omega} = \bar{\omega} = \cos(\theta) - i\sin(\theta)$.

Then $\omega^2 + \dfrac{5}{\omega} - 2 = (\cos(2\theta) + 5\cos(\theta) - 2) + i(\sin(2\theta) - 5\sin(\theta))$

Since $\omega^2 + \dfrac{5}{\omega} - 2$ is purely imaginary, we have $0 = \mathsf{Re}\left(\omega^2 + \dfrac{5}{\omega} - 2\right) = \cos(2\theta) + 5\cos(\theta) - 2 = 2\cos^2(\theta) + 5\cos(\theta) - 3$.

Then $(2\cos(\theta) - 1)(\cos(\theta) + 3) = 0$.

Therefore $\cos(\theta) = \dfrac{1}{2}$ or $\cos(\theta) = -3$. The possibility '$\cos(\theta) = -3$' is rejected. Then $\cos(\theta) = \dfrac{1}{2}$.

Therefore $\sin(\theta) = \dfrac{\sqrt{3}}{2}$ or $\sin(\theta) = -\dfrac{\sqrt{3}}{2}$. Since $\mathsf{Im}(\omega) \geq 0$, we have $\mathsf{Im}(\omega) = \sin(\theta) = \dfrac{\sqrt{3}}{2}$.

Therefore $\omega = \cos(\theta) + i\sin(\theta) = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$.

11. **Answer.**

(a) 1

(b) *Hint.* Be aware that $|\lambda| = 1$ and $\overline{\lambda} = \dfrac{1}{\lambda}$. Also be aware that $\lambda^2 = \cos(2\alpha) + i\sin(2\alpha)$.

(c) ——

12. **Solution.**

(a) i. Let z be a complex number.
Note that $|z|^2 - |\mathsf{Re}(z)|^2 = |z|^2 - (\mathsf{Re}(z))^2 = (\mathsf{Im}(z))^2 \geq 0$.
Then $|z|^2 \geq (|\mathsf{Re}(z)|)^2$. Note that $|z| \geq 0$ and $|\mathsf{Re}(z)| \geq 0$. Therefore $|z| \geq |\mathsf{Re}(z)|$

- Suppose $|z| = |\mathsf{Re}(z)|$. Then $(\mathsf{Im}(z))^2 = 0$. Therefore $\mathsf{Im}(z) = 0$. Hence z is real.
- Suppose z is real. Then $z = \mathsf{Re}(z)$. Therefore $|z| = |\mathsf{Re}(z)|$.

ii. Let z be a complex number.
Define $w = -iz$. We have $w = -i(\mathsf{Re}(z) + i\mathsf{Im}(z)) = \mathsf{Im}(z) - i\mathsf{Re}(z)$. Then $\mathsf{Re}(w) = \mathsf{Im}(z)$ and $\mathsf{Im}(w) = -\mathsf{Re}(z)$.
We have $|z| = |-iz| = |w| \geq |\mathsf{Re}(w)| = |\mathsf{Im}(z)|$.
Equality holds iff w is real. The latter holds iff z is purely imaginary.

(b) Let u, v be complex numbers.
By the result in part (a), $|\mathsf{Re}(u\bar{v})| \leq |u\bar{v}| = |u||\bar{v}| = |u||v|$.

- Suppose $|\mathsf{Re}(u\bar{v})| \leq |u||v|$. Then, by the result in part (a), $u\bar{v}$ is real.
  * (Case 1.) Suppose $v = 0$. Then $0 \cdot u + 1 \cdot v = 0$, and $0, 1 \in \mathbb{R}$ with $1 \neq 0$.
  * (Case 2.) Suppose $v \neq 0$.
    Write $p = |v|^2$, $q = -u\bar{v}$. Note that $p, q \in \mathbb{R}$, and $p \neq 0$.
    We have $pu = |v|^2 u = u\bar{v}v = -qv$. Then $pu + qv = 0$.
- Suppose there exist some $p, q \in \mathbb{R}$ such that $pu + qv = 0$ and p, q are not both zero.
  Without loss of generality, assume $p \neq 0$. Write $r = -q/p$.
  We have $u = rv$. Then $\mathsf{Im}(u\bar{v}) = \mathsf{Im}(rv\bar{v}) = \mathsf{Im}(r|v|^2) = 0$. Therefore $u\bar{v}$ is real. Hence $\mathsf{Re}(u\bar{v}) = |u||v|$.

(c) i. Let z, w be complex numbers.

$$
\begin{aligned}
& (|z| + |w|)^2 - |z + w|^2 \\
= \ & |z|^2 + |w|^2 + 2|z||w| - (z + w)\overline{(z + w)} = |z|^2 + |w|^2 + 2|z||w| - (z + w)(\bar{z} + \bar{w}) \\
& \vdots \\
= \ & |z|^2 + |w|^2 + 2|z||w| - (|z|^2 + |w|^2 + 2\mathsf{Re}(z\bar{w})) \\
= \ & 2(|z||w| - \mathsf{Re}(z\bar{w})) \geq 2(|z||w| - |\mathsf{Re}(z\bar{w})|) \geq 0.
\end{aligned}
$$

Then $|z + w|^2 \leq (|z| + |w|)^2$.
Since $|z + w|, |z|, |w|$ are all non-negative, we have $|z + w| \leq |z| + |w|$.
Note that $|z + w| = |z| + |w|$ iff $(\mathsf{Re}(z\bar{w}) = |\mathsf{Re}(z\bar{w})|$ and $|\mathsf{Re}(z\bar{w})| = |z||w|)$.

- Suppose $|z + w| = |z| + |w|$.
  Then $\mathsf{Re}(z\bar{w}) = |\mathsf{Re}(z\bar{w})|$ and $|\mathsf{Re}(z\bar{w})| = |z||w|$.
  Since $|\mathsf{Re}(z\bar{w})| = |z||w|$, by the result in part (b), there exist some real numbers p, q such that $pz + qw = 0$ and p, q are not both zero.
  Without loss of generality, assume $p \neq 0$.
  Define $s = p^2$ and $t = -pq$. Note that $s > 0$.
  We have $sz = p^2 z = -pqw = tw$.
  Since $\mathsf{Re}(z\bar{w}) = |\mathsf{Re}(z\bar{w})|$, $\mathsf{Re}(z\bar{w})$ is a non-negative real number.
  Now $0 \leq s\mathsf{Re}(z\bar{w}) = \mathsf{Re}(sz\bar{w}) = \mathsf{Re}(tw\bar{w}) = \mathsf{Re}(t|w|^2) = t|w|^2$. Since $|w|^2 \geq 0$, we have $t \geq 0$.

- Suppose there exist some non-negative real numbers s, t such that $sz = tw$ and s, t are not both zero. Without loss of generality, assume $s > 0$.
  We have $z = tw/s$. Then $z\bar{w} = \dfrac{t}{s}|w|^2 > 0$.
  Therefore $\mathsf{Re}(z\bar{w}) = \dfrac{t}{s}|w|^2 = |\mathsf{Re}(z\bar{w})|$.
  Note that $sz - tw = 0$. Then by the result in part (b), we have $|\mathsf{Re}(z\bar{w})| \le |z||w|$.
  Hence $(\mathsf{Re}(z\bar{w}) = |\mathsf{Re}(z\bar{w})|$ and $|\mathsf{Re}(z\bar{w})| = |z||w|)$. Therefore $|z + w| = |z| + |w|$.

ii. Suppose u, v are complex numbers.
We have $|u| = |(u - v) + v| \le |u - v| + |v|$. Then $|u| - |v| \le |u - v|$.
We also have $|v| = |u + (v - u)| \le |u| + |v - u| = |u| + |u - v|$. Then $|u| - |v| \ge -|u - v|$.
Therefore $-|u - v| \le |u| - |v| \le |u - v|$. Hence $|\,|u| - |v|\,| \le |u - v|$.
$|\,|u| - |v|\,| = |u - v|$ iff $(|u| = |u - v| + |v|$ or $|v| = |u| + |v - u|)$.

- Suppose $|\,|u| - |v|\,| = |u - v|$.
  Then $|u| = |u - v| + |v|$ or $|v| = |u| + |v - u|$. Without loss of generality, assume $|u| = |u - v| + |v|$.
  Then there exist some non-negative real numbers s, t such that $s(u - v) = tv$ and s, t are not both zero.
  Define $h = s$ and $k = s + t$. Note that h, k are non-negative real numbers.
  If $s = 0$ then $t > 0$ and $k = s + t > 0$. If $t = 0$ then $h = s > 0$. Hence h, k are not both zero.
  We have $hu = su = (s + t)v = kv$.
- Suppose there exist some non-negative real numbers h, k such that $hu = kv$ and h, k are not both zero.
  Without loss of generality, assume $|u| \ge |v|$.
  We have $h|u| = |hu| = |kv| = k|v|$. Then $h \le k$. Since $h \ge 0$ and $k \ge 0$ and h, k are not both zero, we have $k > 0$.
  Define $s = h$ and $t = k - h$. Then $s > 0$, and $t \ge 0$.
  We have $su = hu = kv = (k - h)v + hv = tv + sv$. Then $s(u - v) = tv$.
  Therefore by the result in part (c.i), we have $|u| = |(u - v) + v| = |u - v| + |v|$. Hence $|u - v| = |u| - |v| = |\,|u| - |v|\,|$.

13. —