

1. **Definition.**

Let  $A$  be a set.

- (1)  $A$  is **countable** if  $A \lesssim \mathbb{N}$ .
- (2)  $A$  is said to be **countably infinite** if  $A \sim \mathbb{N}$ .
- (3)  $A$  is said to be **uncountable** if  $A$  is not countable.

**Basic examples of countably infinite sets.**

- (a)  $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ ;
- (b)  $\mathbb{N}^2, \mathbb{N}^3, \mathbb{N}^4, \dots$  .

**Basic examples of uncountable sets.**

- (a)  $\text{Map}(\mathbb{N}, \{0, 1\}), \text{Map}(\mathbb{N}, \llbracket 0, 9 \rrbracket), \text{Map}(\mathbb{N}, \mathbb{N})$ ;
- (b)  $[0, 1], \mathbb{R}, \mathbb{C}$ ;
- (c)  $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4, \dots$  ;
- (d)  $\mathfrak{P}(\mathbb{N}), \mathfrak{P}(\mathfrak{P}(\mathbb{N})), \mathfrak{P}(\mathfrak{P}(\mathfrak{P}(\mathbb{N}))), \dots$  .

**Theorem (XXI).**

- (1) Suppose  $A$  is a set. Then  $A$  is countable iff ( $A$  is finite or  $A$  is countably infinite).
- (2) Suppose  $A$  is a set. Then  $A$  is countably infinite iff  $A$  is both countable and infinite.
- (3)  $A$  is uncountable iff  $\mathbb{N} < A$ .

**Remark.**

The arguments of the respective statements are word games, possibly involving the application of Schröder-Bernstein Theorem.

**Further remarks.**

- (a) Heuristic idea on ‘being countable’:  $A$  is countable exactly when we can identify  $A$  as a subset of  $\mathbb{N}$  by labeling the elements of  $A$  exhaustively by natural numbers.
- (b) Heuristic idea on ‘being uncountable’:  $A$  is uncountable exactly when there are ‘so many’ elements in  $A$  that there is no way to label all the elements of  $A$  by natural numbers alone.
- (c) We may think of  $\mathbb{N}$  or whatever countably infinite set as a ‘smallest’ infinite set.

**Classification of sets by comparing ‘relative sizes’ with the ‘smallest’ infinite set  $\mathbb{N}$ :**

$A$ is finite. ( $A < \mathbb{N}$ .)	$A$ is infinite. ( $\mathbb{N} \lesssim A$ .)	<i>same as left.</i>
<i>same as above.</i>	$A$ is countably infinite. ( $A \sim \mathbb{N}$ .)	<i>same as below.</i>
<i>same as right.</i>	$A$ is countable. ( $A \lesssim \mathbb{N}$ .)	$A$ is uncountable. ( $\mathbb{N} < A$ .)

2. **‘Hilbert’s Hotel’.**

Every ‘infinite’ set can ‘absorb’ a countably infinite set (or a finite set, which is a set of ‘smaller size’ than  $\mathbb{N}$ ) to form a new set of the ‘same size’ as itself.

**Theorem (XXII). (‘Hilbert’s Hotel’.)**

Let  $A, B$  be sets. Suppose  $A$  is infinite and  $B$  is countable. Further suppose  $A \cap B = \emptyset$ . Then  $A \cup B \sim A$ .

**Proof.** The result follows from Theorem (XXI), Lemma (XXIII) and Lemma (XXIV).

**Lemma (XXIII).**

Let  $A, B$  be sets. Suppose  $A$  is infinite and  $B$  is finite. Further suppose  $A \cap B = \emptyset$ . Then  $A \cup B \sim A$ .

**Proof.**

Let  $A, B$  be sets. Suppose  $A$  is infinite and  $B$  is finite. Further suppose  $A \cap B = \emptyset$ .

There is an injective function from  $\mathbb{N}$  to  $A$ . This injective function defines some infinite sequence  $\{x_n\}_{n=0}^\infty$  in  $A$  with no repeated terms:  $x_m \neq x_n$  whenever  $m \neq n$ .

Write  $S = \{x_n \mid n \in \mathbb{N}\}$ .

Since  $B$  is finite, and  $|B| = N$  for some  $N \in \mathbb{N}$ . Write  $B = \{y_0, y_1, \dots, y_{N-1}\}$ .

*Idea. How to proceed with the construction of a bijective function from  $A$  to  $A \cup B$ ?*

*The table below gives the idea:*

$$\begin{array}{l} A \\ A \cup B \end{array} \left\| \begin{array}{l} S \\ S \cup B \end{array} \right\| \begin{array}{cccc|cccc} x_0 & x_1 & \cdots & x_{N-1} & x_N & x_{N+1} & x_{N+2} & \cdots \\ \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\ y_0 & y_1 & \cdots & y_{N-1} & x_0 & x_{0+1} & x_{0+2} & \cdots \end{array} \left\| \begin{array}{l} z \in A \setminus S \\ z \in A \setminus S \end{array} \right.$$

Define [with the table above in mind]

$$\begin{aligned} F_1 &= \{(x_i, y_i) \mid i \in \llbracket 0, N-1 \rrbracket\} \cup \{(x_j, x_{j-N}) \mid j \in \mathbb{N} \text{ and } j \geq N\}, \\ F_2 &= \{(z, z) \mid z \in A \setminus S\}. \end{aligned}$$

Note that  $F_1 \subset A \times (A \cup B)$  and  $F_2 \subset A \times (A \cup B)$ .

Define the relation  $f = (A, A \cup B, F)$  by  $F = F_1 \cup F_2$ .

$f$  is a bijective function from  $A$  to  $A \cup B$ . (Why?) It follows that  $A \sim A \cup B$ .

**Lemma (XXIV).**

*Let  $A, B$  be sets. Suppose  $A$  is infinite and  $B$  is countably infinite. Further suppose  $A \cap B = \emptyset$ . Then  $A \cup B \sim A$ .*

**Proof.**

Let  $A, B$  be sets. Suppose  $A$  is infinite and  $B$  is countably infinite. Further suppose  $A \cap B = \emptyset$ .

There is an injective function from  $\mathbb{N}$  to  $A$ . This injective function defines some infinite sequence  $\{x_n\}_{n=0}^\infty$  in  $A$  with no repeated terms:  $x_m \neq x_n$  whenever  $m \neq n$ .

Write  $S = \{x_n \mid n \in \mathbb{N}\}$ .

Since  $B \sim \mathbb{N}$ , there is a bijective function from  $\mathbb{N}$  to  $B$ . This bijective function defines some infinite sequence  $\{y_n\}_{n=0}^\infty$  in  $B$ , exhausting  $B$  and with no repeated terms:  $B = \{y_n \mid n \in \mathbb{N}\}$ , and  $y_m \neq y_n$  whenever  $m \neq n$ .

*Idea. How to proceed with the construction of a bijective function from  $A$  to  $A \cup B$ ? This is the idea:*

$$\begin{array}{l} A \\ A \cup B \end{array} \left\| \begin{array}{l} S \\ S \cup B \end{array} \right\| \begin{array}{cccccc|cccccc} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & \cdots & x_{2N} & x_{2N+1} & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \cdots \\ y_0 & x_0 & y_1 & x_1 & y_2 & x_2 & \cdots & y_N & x_N & \cdots \end{array} \left\| \begin{array}{l} z \in A \setminus S \\ z \in A \setminus S \end{array} \right.$$

Define [with the table above in mind]

$$\begin{aligned} G_1 &= \{(x_{2n}, y_n) \mid n \in \mathbb{N}\} \cup \{(x_{2n+1}, x_n) \mid n \in \mathbb{N}\}, \\ G_2 &= \{(z, z) \mid z \in A \setminus S\} \end{aligned}$$

Note that  $G_1 \subset A \times (A \cup B)$  and  $G_2 \subset A \times (A \cup B)$ .

Define the relation  $g = (A, A \cup B, G)$  by  $G = G_1 \cup G_2$ .

$g$  is a bijective function from  $A$  to  $A \cup B$ . (Why?) It follows that  $A \sim A \cup B$ .

### 3. Consequences of ‘Hilbert’s Hotel’.

The conclusion in Theorem (XXII) still holds when the condition ‘ $A \cap B = \emptyset$ ’ is dropped.

**Corollary (XXV).**

*Let  $C, D$  be sets. Suppose  $C$  is infinite and  $D$  is countable. Then  $C \cup D \sim C$ .*

**Proof.** Exercise. (Apply Theorem (XXII).)

**Corollary (XXVI).**

*Let  $C, D$  be sets. Suppose  $D \subset C$ . Also suppose that  $C \setminus D$  is infinite and  $D$  is countable. Then  $C \setminus D \sim C$ .*

**Proof.** Exercise. (Apply Theorem (XXII).)

**Example of application of Corollary (XXVI).**

$\mathbb{R} \setminus \mathbb{Q} \sim \mathbb{R}$ . Also,  $\mathbb{Q} < \mathbb{R} \setminus \mathbb{Q}$ . Justification:

$\{\sqrt[n+2]{2}\}_{n=0}^\infty$  is an infinite sequence with no repeated terms in  $\mathbb{R} \setminus \mathbb{Q}$ . (Why?) Therefore  $\mathbb{R} \setminus \mathbb{Q}$  is infinite. By Corollary (XXVI),  $\mathbb{R} \setminus \mathbb{Q} \sim \mathbb{R}$ . Since  $\mathbb{Q} < \mathbb{R}$ , we also have  $\mathbb{Q} < \mathbb{R} \setminus \mathbb{Q}$ .

**Remark.** There are as many irrational numbers as real numbers. There are ‘much more’ irrational numbers than rational numbers.

#### 4. Countable union of countable sets.

Recall the definition for the notion of generalized union:

- Let  $M$  be a set and  $\{S_n\}_{n=0}^{\infty}$  be an infinite sequence of subsets of  $M$ . The (generalized) union of  $\{S_n\}_{n=0}^{\infty}$  is defined to be the set  $\{x \in M : x \in S_n \text{ for some } n \in \mathbb{N}\}$ . It is denoted by  $\bigcup_{n=0}^{\infty} S_n$ .

#### Theorem (XXVII). (Countability of countable union of countable sets.)

Let  $A$  be a set. Suppose  $\{A_n\}_{n=0}^{\infty}$  is an infinite sequence of countable subsets of  $A$ . Then  $\bigcup_{n=0}^{\infty} A_n$  is countable.

**Remark.** Hence ‘the (generalized) union of countably many countable sets is countable’.

**Proof.**

- *Idea of the argument.*

Write  $B = \bigcup_{n=0}^{\infty} A_n$ .

For each  $n \in \mathbb{N}$ , label the elements of  $A_n$  exhaustively with elements of  $\mathbb{N}$ , so that we have  $A_n = \{x_{n0}, x_{n1}, x_{n2}, \dots\}$ .

Now obtain the possibly ‘infinite array’, which exhausts the elements of the set  $B$ :

$$\begin{array}{c|cccc} A_0 & x_{00} & x_{01} & x_{02} & x_{03} & \cdots \\ A_1 & x_{10} & x_{11} & x_{12} & x_{13} & \cdots \\ A_2 & x_{20} & x_{21} & x_{22} & x_{23} & \cdots \\ A_3 & x_{30} & x_{31} & x_{32} & x_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

For each  $n$ , the ‘row’ of the ‘ $x_{nj}$ ’s may ‘terminate’ or not. There may be ‘repeated’ entries amongst the ‘ $x_{ij}$ ’s’. No matter what, the ‘size’ of this ‘array’ is at most that of  $\mathbb{N}^2$  and hence that of  $\mathbb{N}$ . Hence  $B$  is countable.

- *Formal argument.* Exercise.

#### Corollary (XXVIII). (Sufficiency criteria for being countably infinite.)

Let  $A$  be a set, and  $\{A_n\}_{n=0}^{\infty}$  be an infinite sequence of countable subsets of  $A$ .

- (1) Suppose  $\bigcup_{n=0}^{\infty} A_n$  is infinite. Then  $\bigcup_{n=0}^{\infty} A_n$  is countably infinite.
- (2) Suppose there exists some  $m \in \mathbb{N}$  such that  $A_m$  is countably infinite. Then  $\bigcup_{n=0}^{\infty} A_n$  is countably infinite.
- (3) Suppose there exists some infinite sequence  $\{x_n\}_{n=0}^{\infty}$  in  $A$  such that both of the statements below hold:
  - (3a)  $(x_n \in A_n \text{ for any } n \in \mathbb{N})$  and
  - (3b) (for any  $k, m \in \mathbb{N}$ , if  $k \neq m$  then  $x_k \neq x_m$ ).

Then  $\bigcup_{n=0}^{\infty} A_n$  is countably infinite.

5. Examples of applications of Theorem (XXVII), Theorem (XXVIII).

(1) Another argument for  $\mathbb{Q} \sim \mathbb{N}$ :

- Write  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ .

For any  $n \in \mathbb{N}^*$ , define  $Q_n = \left\{ \frac{m}{n} \mid m \in \mathbb{Z} \right\}$ . We have  $\mathbb{Q} = \bigcup_{n=1}^{\infty} Q_n$ .

Note that  $Q_n \sim \mathbb{Z} \sim \mathbb{N}$ . Then  $\mathbb{Q} \lesssim \mathbb{N}$ .

Recall  $\mathbb{N} \lesssim \mathbb{Q}$ . Then we have  $\mathbb{Q} \sim \mathbb{N}$ .

(2) Denote by  $\mathbb{Q}[x]$  the set of all polynomials with indeterminate  $x$  and with coefficients in  $\mathbb{Q}$ .

$\mathbb{Q}[x] \setminus \{0\} \sim \mathbb{N}$ . Also,  $\mathbb{Q}[x] \sim \mathbb{N}$ . Justification:

- For any  $n \in \mathbb{N}$ , define  $T_n = \{f(x) \in \mathbb{Q}[x] : \deg(f(x)) = n\}$ .

We have  $\mathbb{Q}[x] \setminus \{0\} = \bigcup_{n=0}^{\infty} T_n$ .

Write  $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$ . Note that  $T_n \sim \mathbb{Q}^* \times \mathbb{Q}^n \sim \mathbb{N}$ . Then  $\mathbb{Q}[x] \setminus \{0\} \lesssim \mathbb{N}$ .

For each  $j \in \mathbb{N}$ , denote by  $x^j$  the monic monomial of degree  $j$ .

$\mathbb{N} \sim \{x^j \mid j \in \mathbb{N}\} \lesssim \mathbb{Q}[x] \setminus \{0\}$ . Then we have  $\mathbb{Q}[x] \setminus \{0\} \sim \mathbb{N}$ .

Then  $\mathbb{Q}[x] \sim \mathbb{N}$  also. (Why?)

(3) Define  $\mathbb{A} = \{\zeta \in \mathbb{C} : \zeta \text{ is a root of } f(x) \text{ for some } f(x) \in \mathbb{Q}[x] \setminus \{0\}\}$ .

The elements of  $\mathbb{A}$  are called **algebraic numbers (over  $\mathbb{Q}$ )**. The elements of  $\mathbb{C} \setminus \mathbb{A}$  are called **transcendental numbers (over  $\mathbb{Q}$ )**.

$\mathbb{A} \sim \mathbb{N}$ . Justification:

- For any  $f(x) \in \mathbb{Q}[x] \setminus \{0\}$ , define  $Z[f(x)] = \{\zeta \in \mathbb{C} : \zeta \text{ is a root of } f(x)\}$ . ( $Z[f(x)]$  is the set of all roots of  $f(x)$  in  $\mathbb{C}$ .)

$Z[f(x)]$  is finite, and hence  $Z[f(x)] \lesssim \mathbb{N}$ .

Since  $\mathbb{Q}[x] \setminus \{0\}$  is countably infinite, we may ‘arrange’ all the elements of  $\mathbb{Q}[x] \setminus \{0\}$  as an infinite sequence without repeating terms  $\{f_n(x)\}_{n=0}^{\infty}$ , so that  $\mathbb{Q}[x] \setminus \{0\} = \{f_n(x) \mid n \in \mathbb{N}\}$ . (How? Why?)

Now  $\mathbb{A} = \bigcup_{n=0}^{\infty} Z[f_n(x)]$ . Then  $\mathbb{A} \lesssim \mathbb{N}$ .

Note that  $\mathbb{N} \lesssim \mathbb{A}$ . Therefore we have  $\mathbb{A} \sim \mathbb{N}$ .

**Remark.** It follows that  $\mathbb{A} < \mathbb{C}$  and  $\mathbb{A} \cap \mathbb{R} < \mathbb{R}$ . There are real/complex numbers which are not algebraic; actually there are much more (real/complex) transcendental numbers than there are (real/complex) algebraic numbers. This is a non-constructive proof of the existence of transcendental numbers.

6.  $\mathbb{N}$  is the ‘smallest’ infinite set.

By Cantor’s Theorem, we have  $\mathbb{N} < \mathfrak{P}(\mathbb{N}) < \mathfrak{P}(\mathfrak{P}(\mathbb{N})) < \dots$ .

Question (1).

*Is there a set of cardinality greater than each of  $\mathbb{N}$ ,  $\mathfrak{P}(\mathbb{N})$ ,  $\mathfrak{P}(\mathfrak{P}(\mathbb{N}))$ , ...?*

Answer to Question (1).

Yes, one such set is the ‘union’ of all these sets. To make sense of this set, we need the Axiom of Substitution.

Question (2).

*Is there a set of cardinality greater than  $\mathbb{N}$  and less than  $\mathbb{R}$ ?*

Answer to Question (2).

Cantor believed there was no such set.

**Cantor’s Continuum Hypothesis.**

*For any set  $S$ , if  $\mathbb{N} \lesssim S \lesssim \mathbb{R}$  then  $(S \sim \mathbb{N} \text{ or } S \sim \mathbb{R})$ .*

So what are  $\mathbb{N}$  and  $\mathbb{R}$ , really? Or, what is the respective nature of these two sets? This leads us to the foundation of mathematics.