

1. Definition.

Let A be a set.

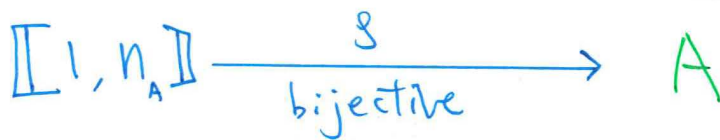
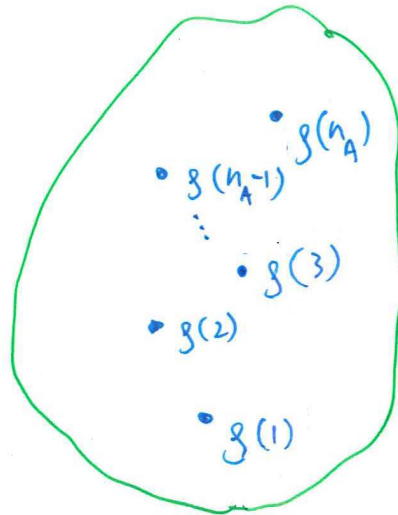
A is said to be **finite** if there exists some $n_A \in \mathbf{N}$ such that $A \sim \llbracket 1, n_A \rrbracket$.

The number n_A is called the cardinality of A .

Remark.

The number n_A is uniquely determined by A . (Proof?)

$\exists n_A \in \mathbf{N}$,
 $g: \llbracket 1, n_A \rrbracket \rightarrow A$
such that
 g is bijective.



$$A = \{g(1), g(2), g(3), \dots, g(n_A)\}$$

For any $k, m \in \llbracket 1, n_A \rrbracket$,
if $k \neq m$ then $g(k) \neq g(m)$.

So A has exactly n_A elements.

Convention: $\llbracket 1, 0 \rrbracket = \emptyset$.

Hence $n_A = 0$ iff $A = \emptyset$.

2. Illustration of properties of finite sets through an example.

Consider the set $\{1, 2, 3\}$.

(1) *Question.*

Is there some injective function from \mathbf{N} to $\{1, 2, 3\}$? Is there an infinite sequence with no repeated terms in $\{1, 2, 3\}$?

Answer and reason.

No. Were there an infinite sequence $(a_n)_{n=0}^{\infty}$ with no repeated terms in $\{1, 2, 3\}$, we would have $\{a_0, a_1, a_2\} = \{1, 2, 3\}$ and then a_3 would have to be one of a_0, a_1, a_2 .

(2) *Question.*

Is there some proper subset U of $\{1, 2, 3\}$ satisfying $\{1, 2, 3\} \sim U$?

Answer and reason.

No. (Heuristic argument.) Every proper subset of $\{1, 2, 3\}$ has at most two elements, but $\{1, 2, 3\}$ has three elements. They are not ‘of the same size’.

(3) *Question.*

Is there some function $\varphi : \{1, 2, 3\} \longrightarrow \{1, 2, 3\}$ which is injective and not surjective?

Answer and reason.

No. There are six injective functions from $\{1, 2, 3\}$ to $\{1, 2, 3\}$. They are given by:

$$\begin{array}{ll} \varphi_1(1) = 1, \varphi_1(2) = 2, \varphi_1(3) = 3. & \varphi_4(1) = 1, \varphi_4(2) = 3, \varphi_4(3) = 2. \\ \varphi_2(1) = 2, \varphi_2(2) = 3, \varphi_2(3) = 1. & \varphi_5(1) = 2, \varphi_5(2) = 1, \varphi_5(3) = 3. \\ \varphi_3(1) = 3, \varphi_3(2) = 1, \varphi_3(3) = 2. & \varphi_6(1) = 3, \varphi_6(2) = 2, \varphi_6(3) = 1. \end{array}$$

They are all surjective.

(4) *Question.*

Is there some function $\psi : \{1, 2, 3\} \longrightarrow \{1, 2, 3\}$ which is surjective and not injective?

Answer and reason.

No. There are six surjective functions from $\{1, 2, 3\}$ to $\{1, 2, 3\}$. They are given by:

$$\begin{array}{ll} \psi_1(1) = 1, \psi_1(2) = 2, \psi_1(3) = 3. & \psi_4(1) = 1, \psi_4(2) = 3, \psi_4(3) = 2. \\ \psi_2(1) = 2, \psi_2(2) = 3, \psi_2(3) = 1. & \psi_5(1) = 2, \psi_5(2) = 1, \psi_5(3) = 3. \\ \psi_3(1) = 3, \psi_3(2) = 1, \psi_3(3) = 2. & \psi_6(1) = 3, \psi_6(2) = 2, \psi_6(3) = 1. \end{array}$$

They are all injective.

We can ask the same questions for every other finite set, and obtain the same answers along the same line of reasoning.

3. **Theorem (XIX).** (**Characterization of finite sets.**)

Let A be a set. The statements below are equivalent:

- (1) *A is finite.*
- (2) *No proper subset of A is of cardinality equal to A .*
- (3) *For any function φ from A to A , if φ is injective then φ is surjective.*
- (4) *For any function ψ from A to A , if ψ is surjective then ψ is injective.*

Proof of Theorem (XIX). A very tedious exercise in mathematical induction.

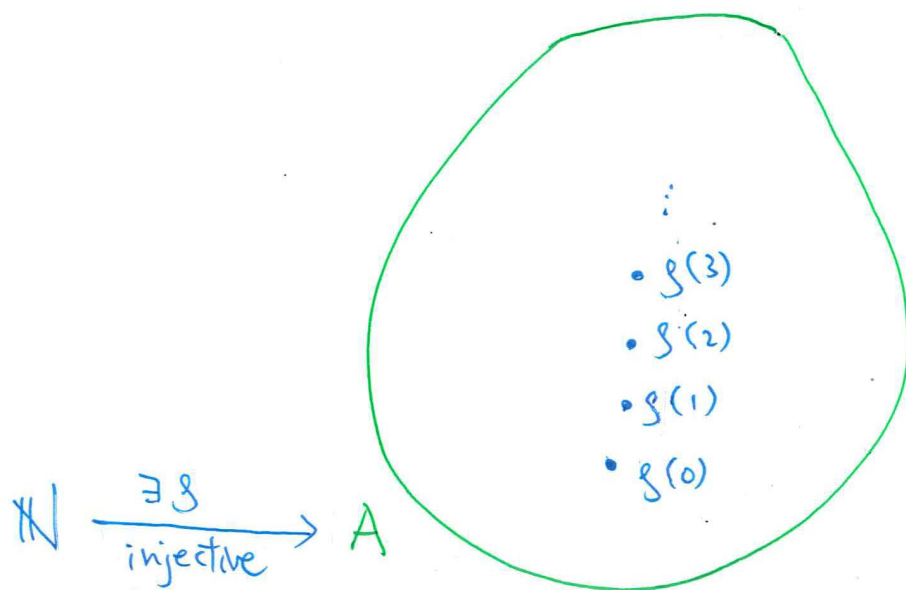
4. Definition.

Let A be a set.

A is said to be **infinite** if $\mathbb{N} \lesssim A$.

Remark.

Heuristic idea in this definition: A is infinite iff A contains at least a 'copy' of \mathbb{N} as a subset.



$g(\mathbb{N}) = \{g(0), g(1), g(2), g(3), \dots\}$
is a 'copy' of \mathbb{N}
in the sense that
 $x \mapsto g(x)$ for any $x \in \mathbb{N}$
defines a bijective function
from \mathbb{N} to $g(\mathbb{N})$.

Examples.

\mathbb{N} , \mathbb{Z} , \mathbb{Q} , \mathbb{R} , \mathbb{C} , $[0, 1]$, line segments, circles, squares, discs in the plane \mathbb{R}^2 , ...

5. Illustration of properties of infinite sets through an example.

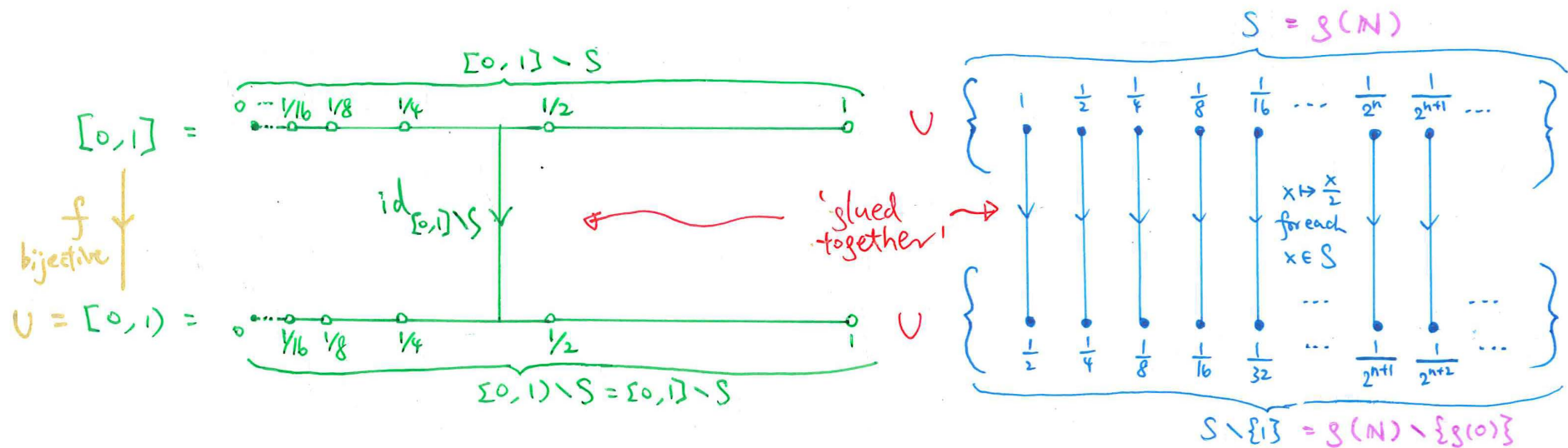
Consider the set $[0, 1]$.

(1) *Question.*

Is there some injective function from \mathbf{N} to $[0, 1]$? Is there an infinite sequence with no repeated terms in $[0, 1]$?

Answer and reason.

Yes. One such infinite sequence is $\{1/2^n\}_{n=0}^{\infty}$. So an injective function g from \mathbf{N} to $[0, 1]$ is given by $g(n) = 1/2^n$ for any $n \in \mathbf{N}$.



(2) *Question.*

Is there some proper subset U of $[0, 1]$ satisfying $[0, 1] \sim U$?

Answer and reason.

Yes. One such set is $[0, 1)$.

(3) Question.

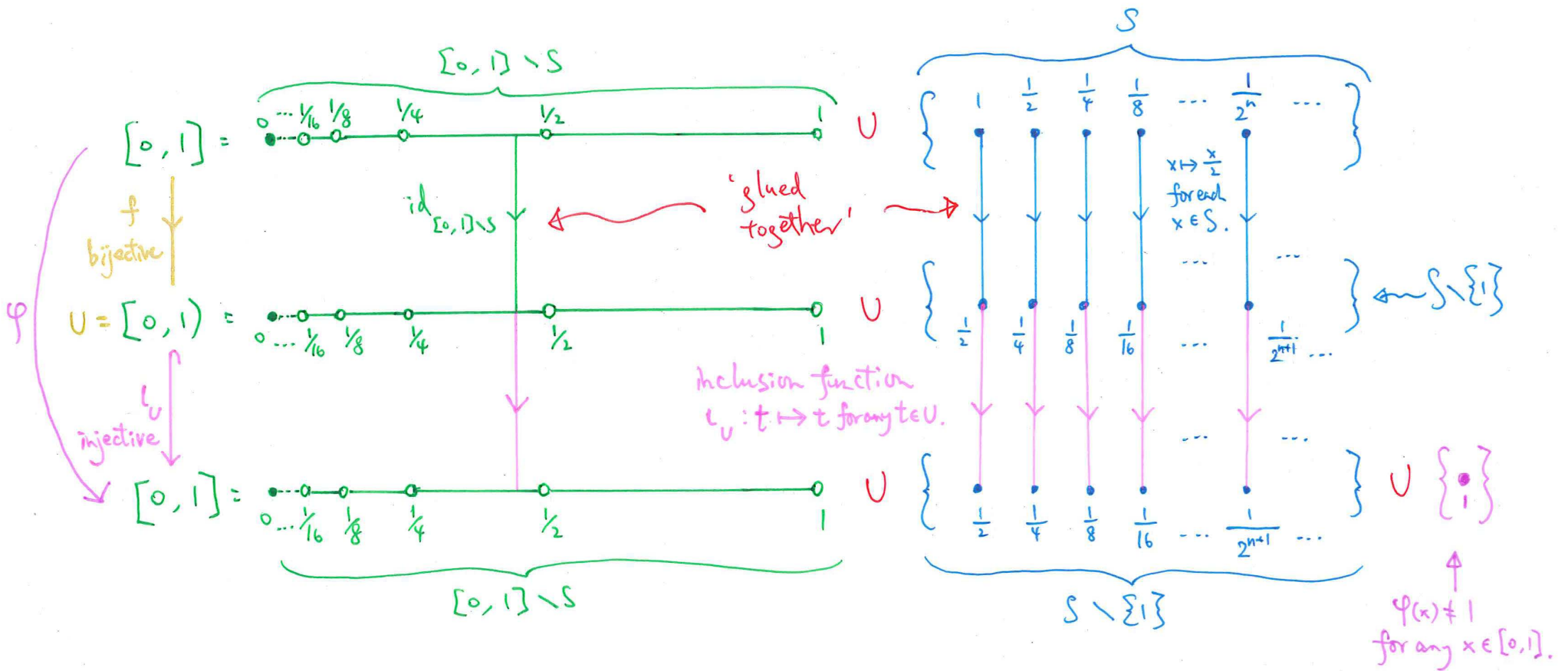
Is there some function $\varphi : [0, 1] \rightarrow [0, 1]$ which is injective and not surjective?

Answer and reason.

Yes. One such function is given by $\varphi : [0, 1] \rightarrow [0, 1]$, where $S = \{1/2^n \mid n \in \mathbf{N}\}$ and

$$\varphi(x) = \begin{cases} x & \text{if } x \in [0, 1] \setminus S \\ x/2 & \text{if } x \in S \end{cases}$$

(Note that for any $x \in [0, 1]$, we have $\varphi(x) \neq 1$.)



(4) *Question.*

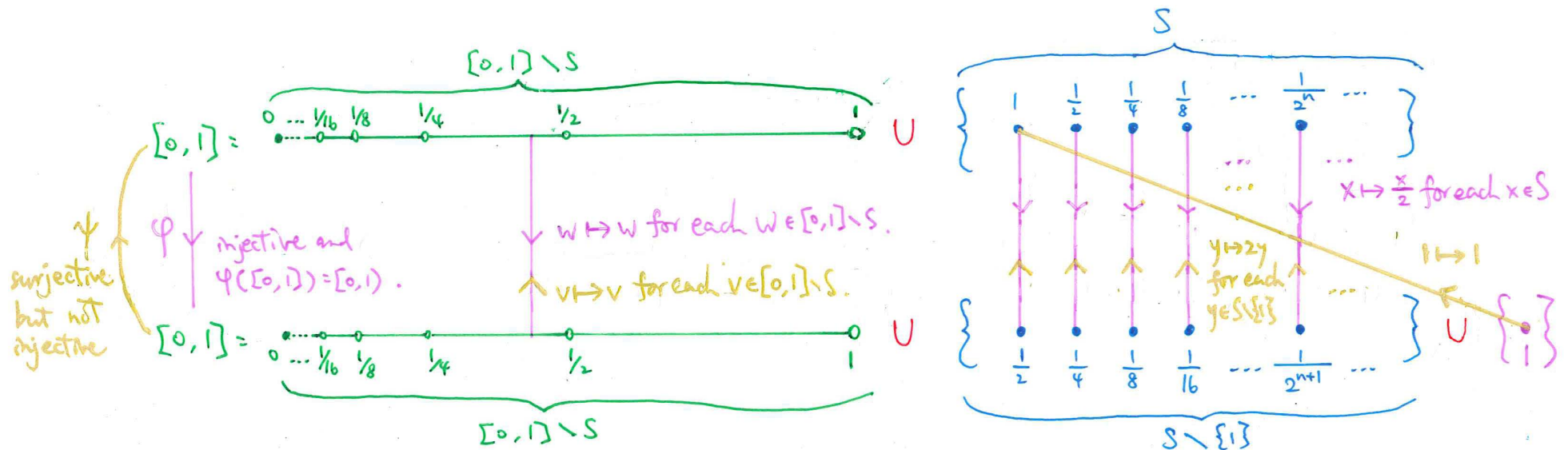
Is there some function $\psi : [0, 1] \longrightarrow [0, 1]$ which is surjective and not injective?

Answer and reason.

Yes. One such function is given by $\psi : [0, 1] \longrightarrow [0, 1]$, where $S = \{1/2^n \mid n \in \mathbf{N}\}$ and

$$\psi(x) = \begin{cases} x & \text{if } x \in [0, 1] \setminus S \\ 2x & \text{if } x \in S \setminus \{1\} \\ 1 & \text{if } x = 1 \end{cases} .$$

(Note that we have $\psi(1) = \psi(1/2) = 1$.)



We can ask the same questions for every other infinite set, and obtain the same answers along the same line of reasoning.

6. Theorem (XX). (Characterization of infinite sets.)

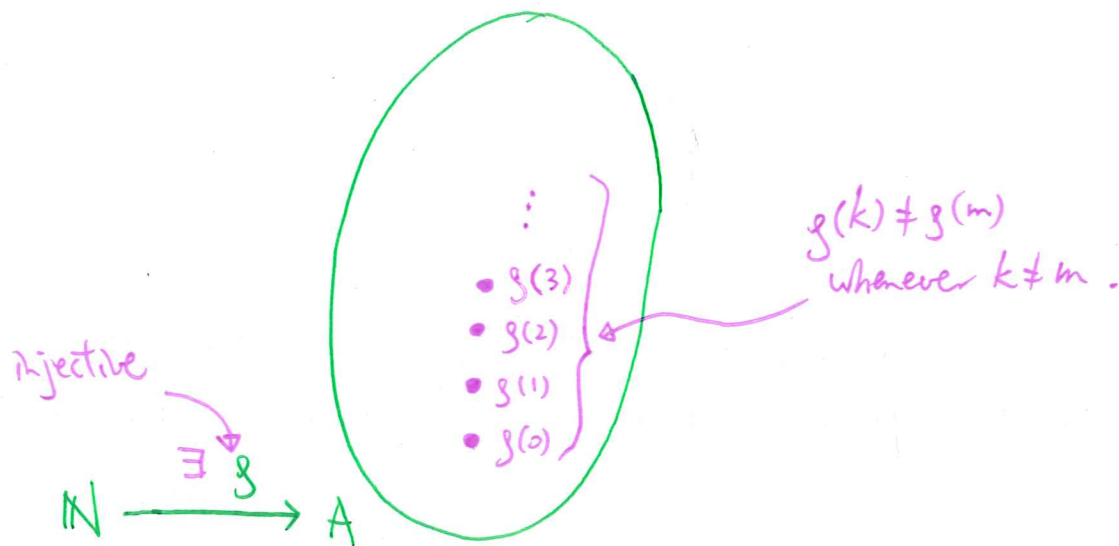
Let A be a set. The statements below are equivalent:

- (1) *A is infinite. ($\mathbf{N} \lesssim A$.)*
- (1') *There exists some subset S of A such that $\mathbf{N} \sim S$.*
- (1'') *There exists some subset T of A such that $\mathbf{N} \lesssim T$.*
- (2) *There exists some proper subset U of A such that $A \sim U$.*
- (2') *There exists some proper subset V of A such that $A \lesssim V$.*
- (3) *There exists some function φ from A to A such that φ is injective and φ is not surjective.*
- (4) *There exists some function ψ from A to A such that ψ is surjective and ψ is not injective.*

Remark.

By Theorem (XIX) and Theorem (XX), every set is finite or infinite, but not both.

Assumption:
 A is infinite.

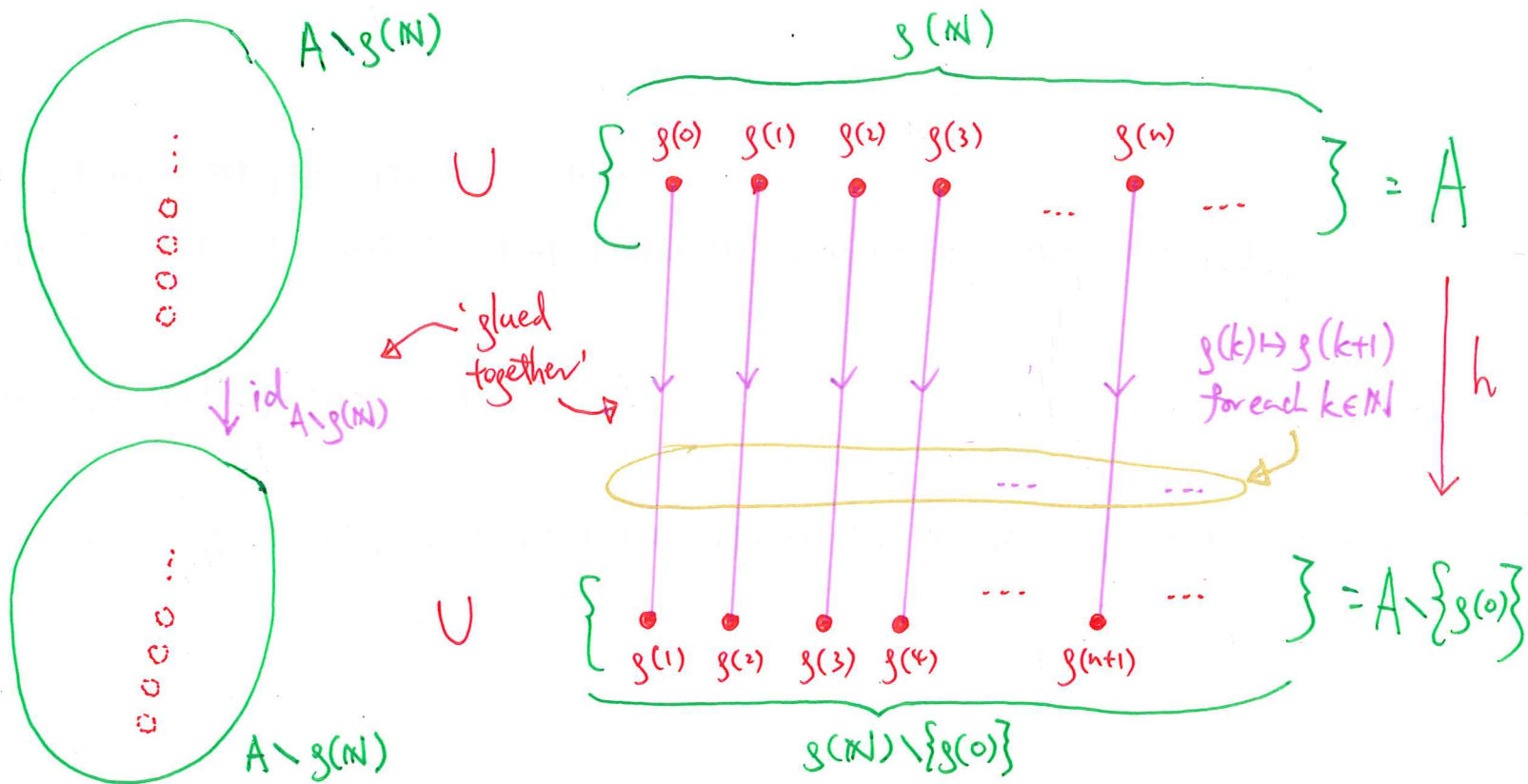


Conclusion.

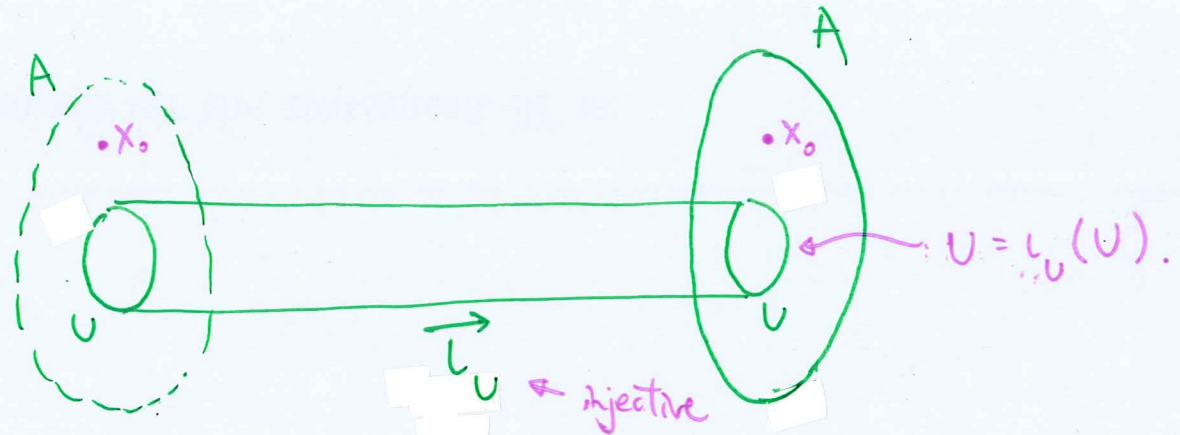
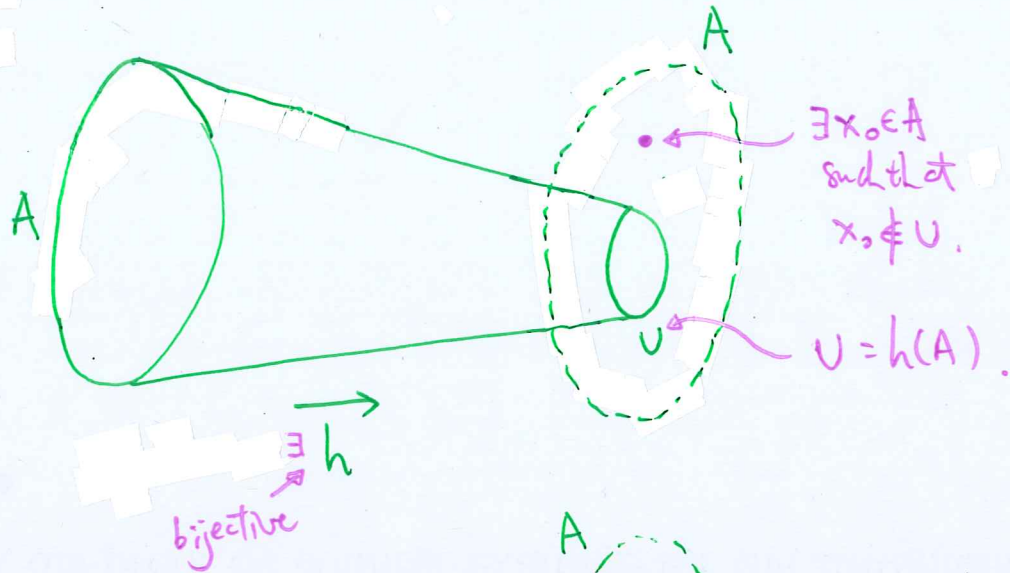
The function
 $h: A \rightarrow A \setminus \{g(0)\}$
 defined by

$$h(x) = \begin{cases} x & \text{if } x \notin g(\mathbb{N}) \\ g(n+1) & \text{if } x = g(n) \text{ for some } n \in \mathbb{N} \end{cases}$$

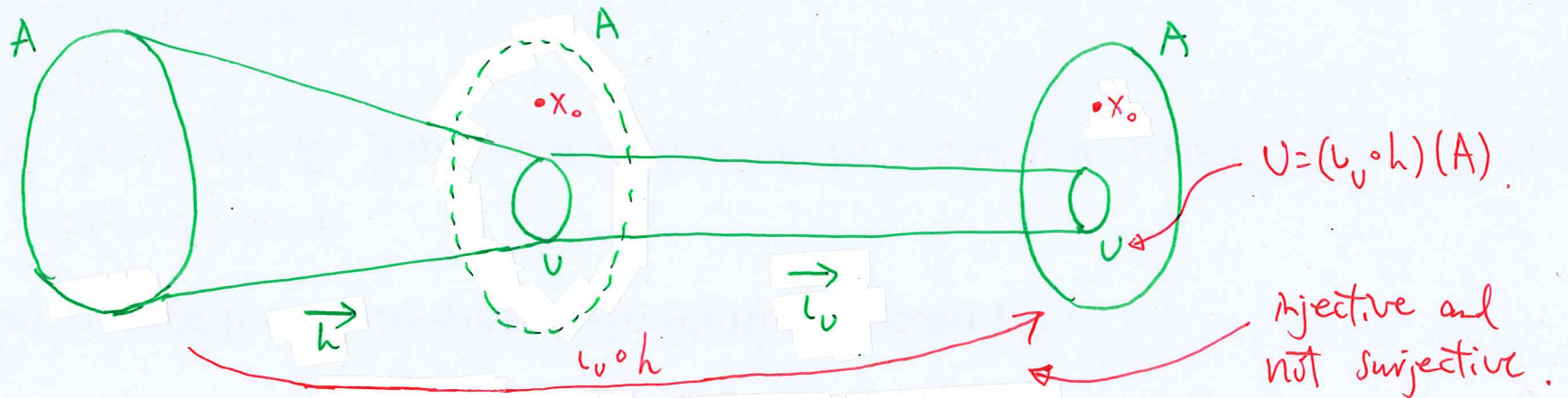
is a bijective function.



Assumption.
 There exists
 some subset U
 of A such that
 $S \cap U$
 $\begin{cases} A \cap U \\ U \neq A. \end{cases}$



Conclusion.



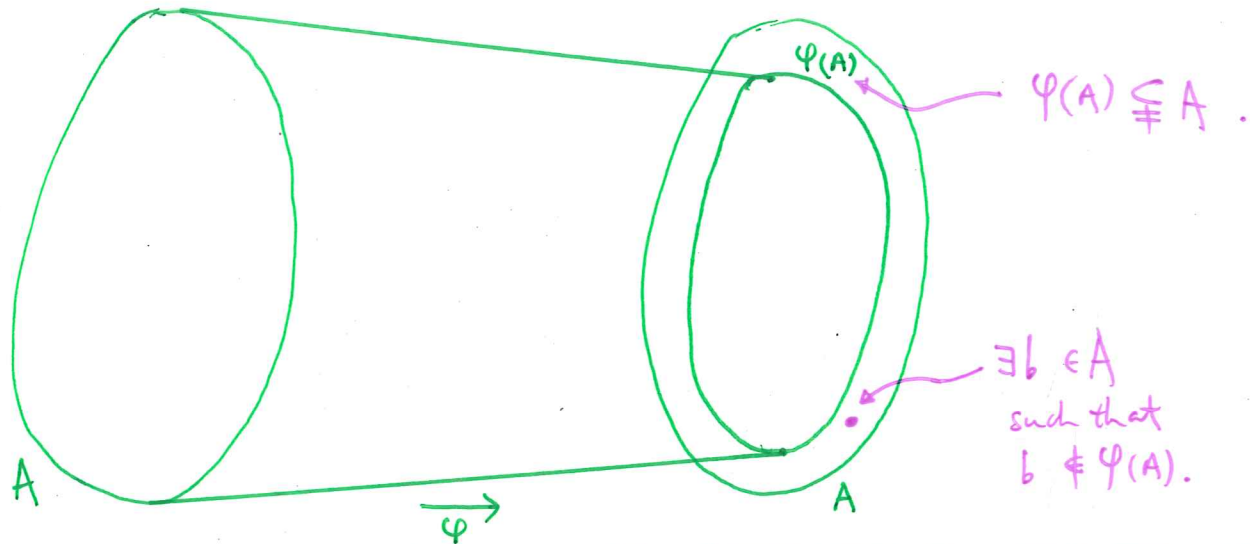
Assumption.

There exists some function

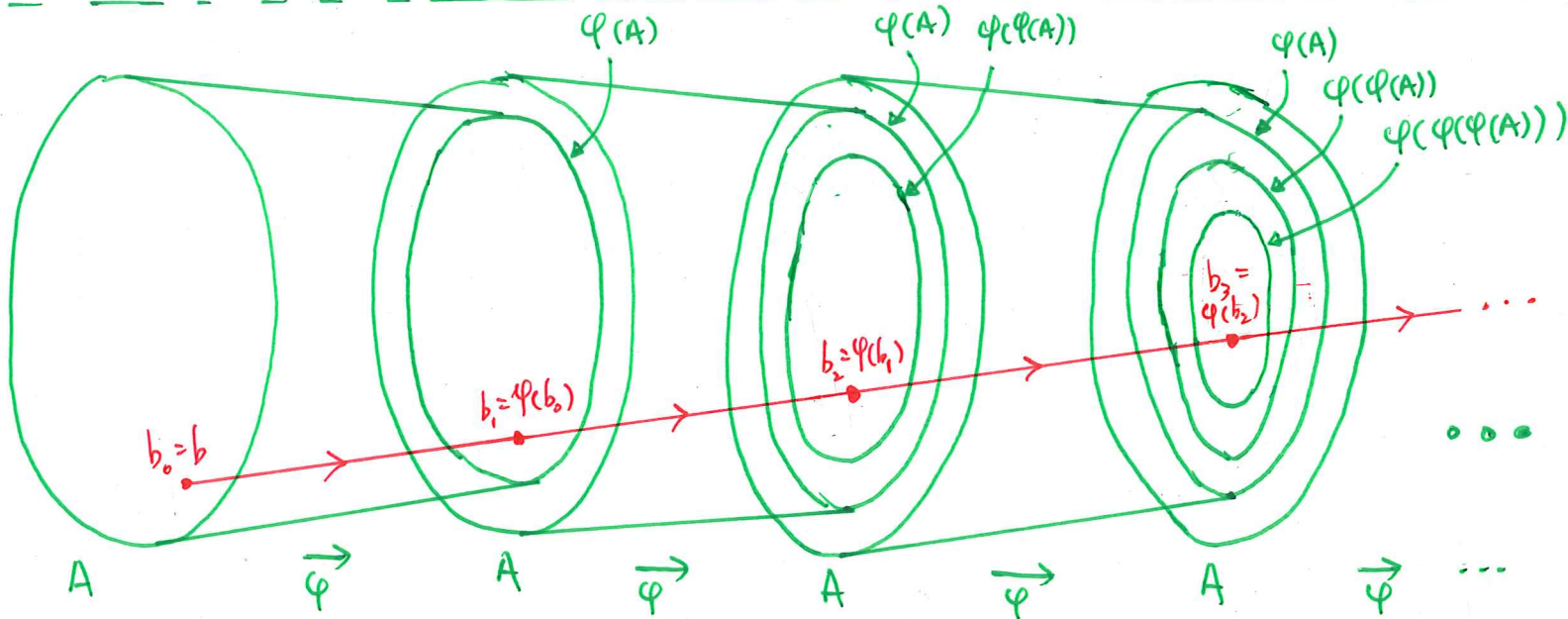
$$\varphi: A \rightarrow A \text{ such that}$$

φ is injective and

φ is not surjective.



Conclusion.



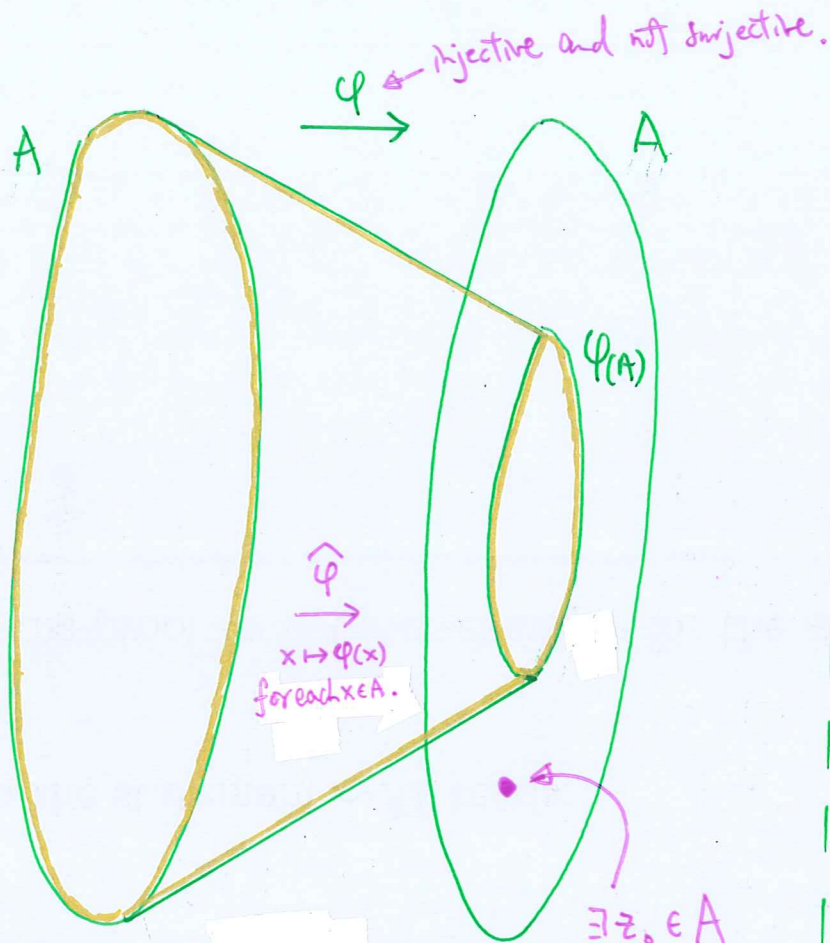
The infinite sequence $\{b_n\}_{n=0}^{\infty}$ defined 'inductively' by $b_n = \begin{cases} b & \text{if } n=0 \\ \varphi(b_{n-1}) & \text{if } n \geq 1 \end{cases}$ is an infinite sequence in A with no repeated terms.

(In fact, for each $m \in \mathbb{N}$, $b_m = (\underbrace{\varphi \circ \varphi \circ \dots \circ \varphi}_{m \text{ times}})(b).$)

The function $f: \mathbb{N} \rightarrow A$ defined by $f(n) = b_n$ for any $n \in \mathbb{N}$ is an injective function. Hence $\mathbb{N} \lesssim A$.

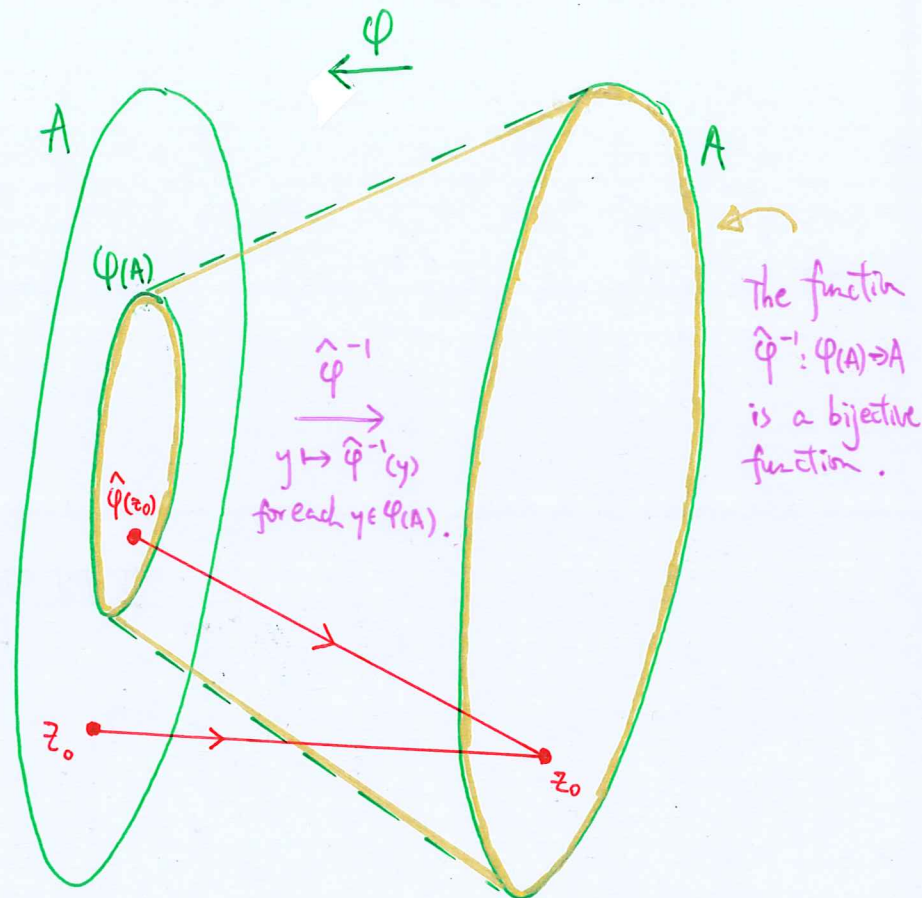
Assumption.

There exists some function $\varphi: A \rightarrow A$ such that φ is injective and φ is not surjective.



The function $\hat{\varphi}: A \rightarrow \varphi(A)$ defined by $\hat{\varphi}(x) = \varphi(x)$ for any $x \in A$ is a bijective function.

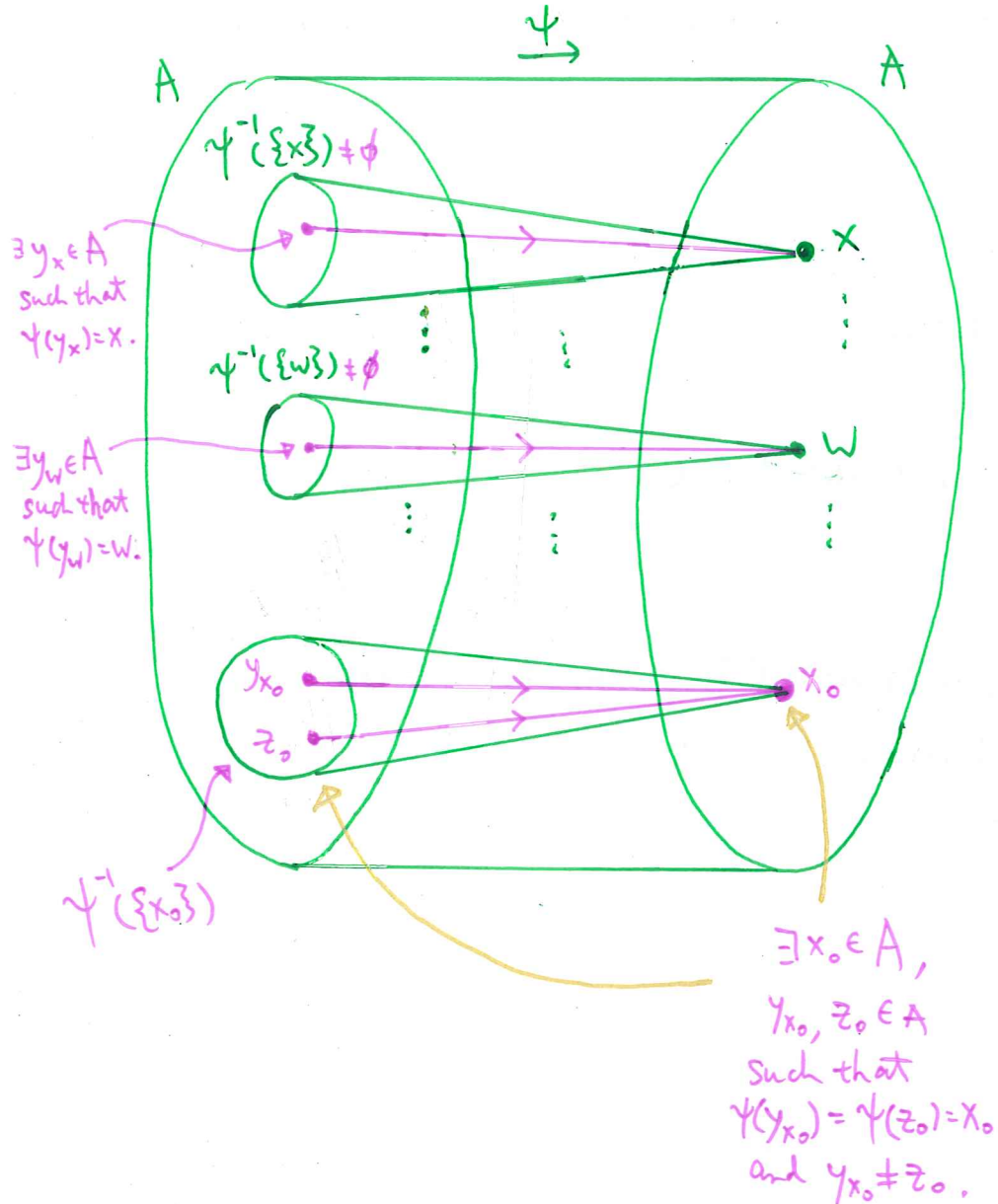
Conclusion.



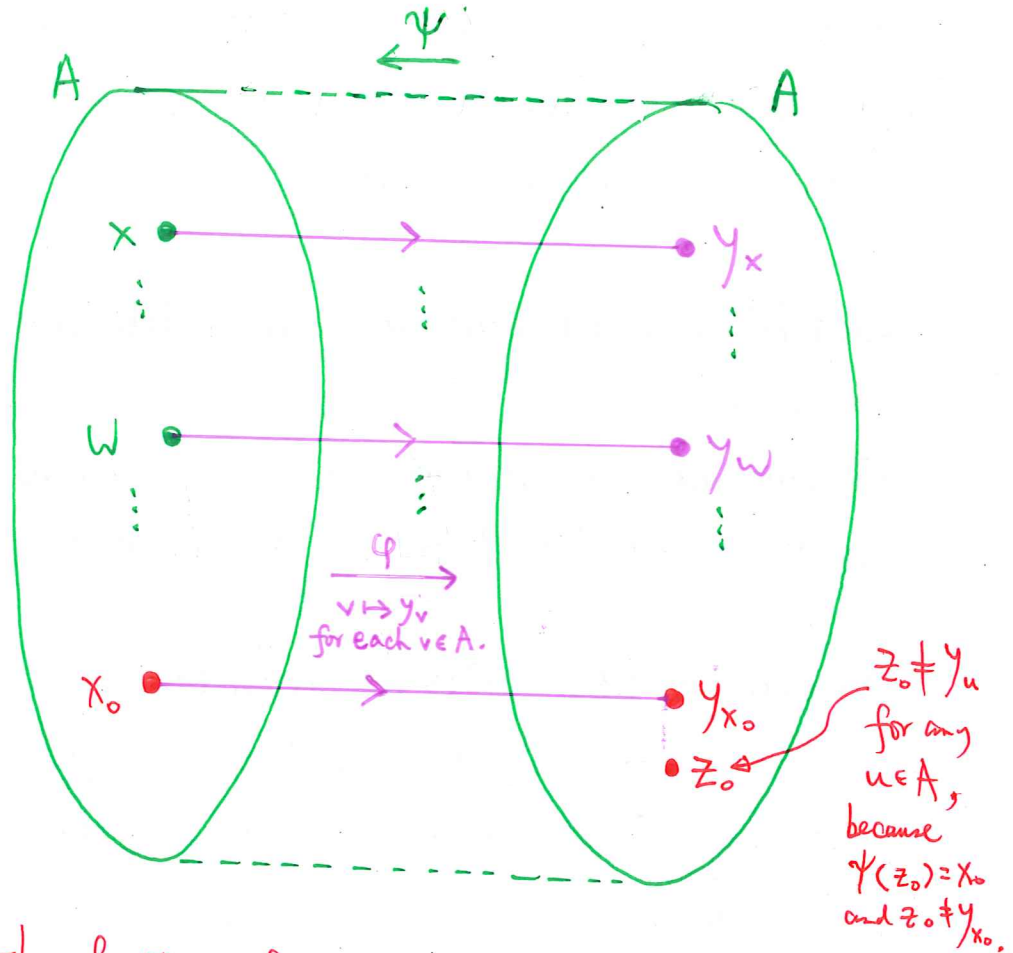
The function $\psi: A \rightarrow A$ defined by
$$\psi(y) = \begin{cases} \hat{\varphi}^{-1}(y) & \text{if } y \in \varphi(A) \\ z_0 & \text{if } y \notin \varphi(A) \end{cases}$$
 is surjective but not injective.

Assumption.

There exists some function $\psi: A \rightarrow A$ such that ψ is surjective and ψ is not injective.



Conclusion.



The function $\phi: A \rightarrow A$ defined by $\phi(x) = y_x$ for any $x \in A$ is injective and not surjective.