

1. Recall:

(a) **Definition.**

Let A, B be sets.

*The set $\text{Map}(A, B)$ is defined to be the **set of all functions from A to B .***

Remark. $\text{Map}(\mathbf{N}, B)$ is the set of all infinite sequences in B : each $\varphi \in \text{Map}(\mathbf{N}, B)$ is the infinite sequence $(\varphi(0), \varphi(1), \varphi(2), \dots, \varphi(n), \varphi(n+1), \dots)$.

(b) **Example** (ϵ).

Let A be a set. $\mathfrak{P}(A) \sim \text{Map}(A, \{0, 1\})$.

(c) **Theorem (VI).**

There is no surjective function from \mathbf{N} to $\text{Map}(\mathbf{N}, \{0, 1\})$.

(d) **Corollary (VII).**

There is no bijective function from \mathbf{N} to $\text{Map}(\mathbf{N}, \{0, 1\})$. (Hence $\mathbf{N} \not\sim \text{Map}(\mathbf{N}, \{0, 1\})$.)

(e) **Theorem (VIII).**

Let A be a set. $A \not\sim \text{Map}(A, \{0, 1\})$. $A \not\sim \mathfrak{P}(A)$.

2. Theorem (XIII). (Baby version of Cantor's Theorem.)

$\mathbb{N} < \text{Map}(\mathbb{N}, \{0, 1\})$.

Proof. [Want to verify: ① $\mathbb{N} \not\sim \text{Map}(\mathbb{N}, \{0, 1\})$. ② $\mathbb{N} \lesssim \text{Map}(\mathbb{N}, \{0, 1\})$.]

By Corollary (VII), $\mathbb{N} \not\sim \text{Map}(\mathbb{N}, \{0, 1\})$.

We now prove that $\mathbb{N} \lesssim \text{Map}(\mathbb{N}, \{0, 1\})$:

- For any $n \in \mathbb{N}$, define $\delta_n : \mathbb{N} \rightarrow \{0, 1\}$ by

$$\delta_n(k) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

← The infinite sequence δ_n is explicitly given by

$$\delta_n = (\underbrace{0, \dots, 0}_{\text{all 0's}}, \underbrace{1}_{n^{\text{th}} \text{ position}}, \underbrace{0, 0, \dots}_{\text{all 0's}}).$$

Define $\Delta : \mathbb{N} \rightarrow \text{Map}(\mathbb{N}, \{0, 1\})$ by $\Delta(n) = \delta_n$ for any $n \in \mathbb{N}$.

Δ is an injective function. (Why?)

Hence $\mathbb{N} \lesssim \text{Map}(\mathbb{N}, \{0, 1\})$.

We now have $\mathbb{N} \lesssim \text{Map}(\mathbb{N}, \{0, 1\})$ and $\mathbb{N} \not\sim \text{Map}(\mathbb{N}, \{0, 1\})$.

It follows that $\mathbb{N} < \text{Map}(\mathbb{N}, \{0, 1\})$.

3. Theorem (XIV). (Cantor's Theorem.)

Suppose A is a set. Then $A < \mathbf{Map}(A, \{0, 1\})$, and $A < \mathfrak{P}(A)$.

Proof.

Let A be a set. By Theorem (VIII), $A \not\sim \mathbf{Map}(A, \{0, 1\})$.

We prove that $A \lesssim \mathbf{Map}(A, \{0, 1\})$:

- Recall that for any $x \in A$, the function $\chi_{\{x\}}^A : A \longrightarrow \{0, 1\}$ is given by

$$\chi_{\{x\}}^A(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

Define the function $\Delta : A \longrightarrow \mathbf{Map}(A, \{0, 1\})$ by $\Delta(x) = \chi_{\{x\}}^A$ for any $x \in A$.

Δ is an injective function from A to $\mathbf{Map}(A, \{0, 1\})$. (Why?)

Hence $A \lesssim \mathbf{Map}(A, \{0, 1\})$.

We now have $A \lesssim \mathbf{Map}(A, \{0, 1\})$ and $A \not\sim \mathbf{Map}(A, \{0, 1\})$.

It follows that $A < \mathbf{Map}(A, \{0, 1\})$.

Since $\mathfrak{P}(A) \sim \mathbf{Map}(A, \{0, 1\})$, we have $A < \mathfrak{P}(A)$. (Why?)

4. Question. Note that $\mathbb{Q} \lesssim \mathbb{R}$. Is it true that $\mathbb{Q} \sim \mathbb{R}$, or that $\mathbb{Q} < \mathbb{R}$?

Lemma (XV).

Let A, B, C be sets. Suppose $A \lesssim B$ and $B \lesssim C$. Also suppose $A < B$ or $B < C$. Then $A < C$.

Proof. [Needed in the argument: Schröder-Bernstein Theorem. 'Let H, K be sets. Suppose $H \lesssim K$ and $K \lesssim H$. Then $H \sim K$!']

Let A, B, C be sets. Suppose $A \lesssim B$ and $B \lesssim C$. Also suppose $A < B$ or $B < C$.

Since $A \lesssim B$ and $B \lesssim C$, we have $A \lesssim C$.

Since $A < B$ or $B < C$, we have $A \not\sim B$ or $B \not\sim C$. We verify that $A \not\sim C$:

- Suppose it were true that $A \sim C$. Then $C \lesssim A$. [We try to deduce $A \sim B$ and $B \sim C$.]

Since $B \lesssim C$ and $C \lesssim A$, we would have $B \lesssim A$.

Then, since $A \lesssim B$ and $B \lesssim A$, we would have $A \sim B$. (Why?)

Since $C \lesssim A$ and $A \lesssim B$, we would have $C \lesssim B$.

Then, since $B \lesssim C$ and $C \lesssim B$, we would have $B \sim C$. (Why?)

Hence $A \sim B$ and $B \sim C$. But by assumption, $A \not\sim B$ or $B \not\sim C$. Contradiction arises.

Hence $A \not\sim C$ in the first place.

Then, since $A \lesssim C$ and $A \not\sim C$, we have $A < C$.

Question. Note that $\mathbb{Q} \lesssim \mathbb{R}$. Is it true that $\mathbb{Q} \sim \mathbb{R}$, or that $\mathbb{Q} < \mathbb{R}$?

Lemma (XV).

Let A, B, C be sets. Suppose $A \lesssim B$ and $B \lesssim C$. Also suppose $A < B$ or $B < C$. Then $A < C$.

Theorem (XVI).

$\mathbb{N} < [0, 1]$, and $\mathbb{N} < \mathbb{R}$, and $\mathbb{Q} < \mathbb{R}$.

Proof.

$\mathbb{N} \lesssim \text{Map}(\mathbb{N}, \{0, 1\}) \lesssim \text{Map}(\mathbb{N}, [0, 9]) \sim [0, 1] \sim \mathbb{R}$.

Also, $\mathbb{N} < \text{Map}(\mathbb{N}, \{0, 1\})$.

Then, by Lemma (XV), $\mathbb{N} < [0, 1]$ and $\mathbb{N} < \mathbb{R}$.

Since $\mathbb{Q} \sim \mathbb{N}$, we also have $\mathbb{Q} < \mathbb{R}$.

Remark.

Hence there are much much more real numbers than there are rational numbers.

5. **Question.** *Why are ‘Venn diagram arguments’ not good enough?*

Theorem (XVII.)

There exists some set T such that $S < T$ for any subset S of \mathbb{R}^2 .

Proof.

Define $T = \mathfrak{P}(\mathbb{R})$.

Pick any subset S of \mathbb{R}^2 . We have $S \lesssim \mathbb{R}^2 \sim \mathbb{R}$.

By Cantor’s Theorem, $\mathbb{R} < \mathfrak{P}(\mathbb{R}) = T$.

Then by Lemma (XV), we have $S < T$.

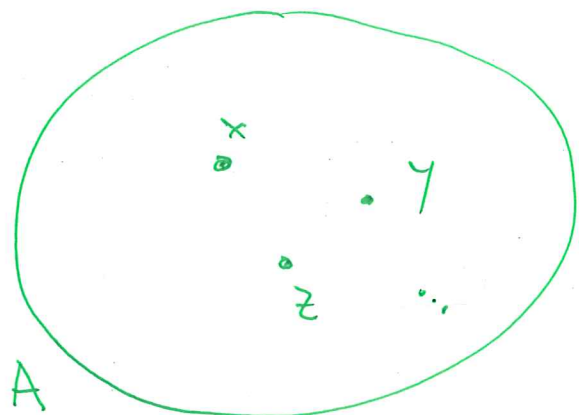
Remark.

When we draw a Venn diagram for a set, say, A , we are ‘identifying’ the set A with some subset, say, B , of \mathbb{R}^2 , in the sense that the elements of A are ‘identified’ as the points in B , via some bijective function from A to B .

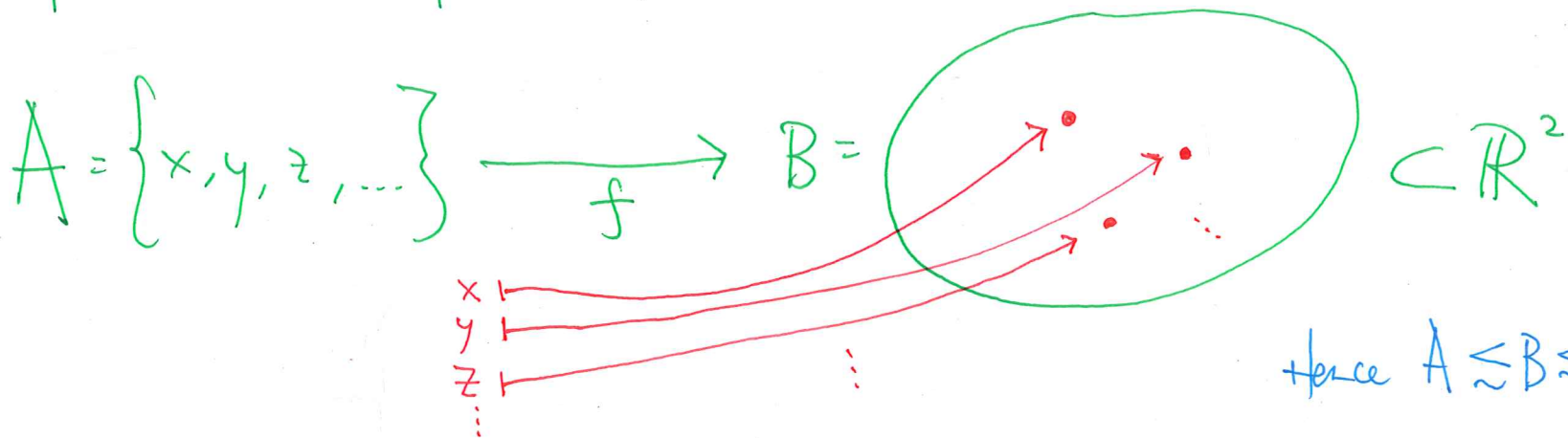
This bijective function guarantees that distinct elements of A are identified as distinct points of B . So we are implicitly assuming that there is an injective function from A to \mathbb{R}^2 .

But now we know that there are sets which are too ‘large’ to be draw in a Venn diagram.

Question: What is implicit in such a 'Venn diagram' of the set $A = \{x, y, z, \dots\}$?



Answer: We are implicitly assuming that there is an injective function f from the set A to some subset B of \mathbb{R}^2 (on which we present the picture):



hence $A \approx B \approx \mathbb{R}$.

6. Question.

Is there any ‘universal set’, which contains every conceivable object as its element?

Theorem (XVIII).

Denote $\{x \mid x = x\}$ by U . The mathematical object U is not a set.

Proof.

Suppose U were a set. Then, by Cantor’s Theorem, $U < \mathbf{Map}(U, \{0, 1\})$.

For any $\varphi \in \mathbf{Map}(U, \{0, 1\})$, we would have $\varphi = \varphi$, and hence $\varphi \in U$.

It would follow that $\mathbf{Map}(U, \{0, 1\}) \subset U$.

Then $\mathbf{Map}(U, \{0, 1\}) \lesssim U$. Therefore $U < \mathbf{Map}(U, \{0, 1\}) \lesssim U$.

By Lemma (XV), $U < U$. In particular, $U \not\sim U$. There would be no bijective function from U to U .

But \mathbf{id}_U is a bijective function from U to U . Contradiction arises.

Hence U is not a set in the first place.

Remark.

Hence if we insist Cantor’s Theorem to be a true statement, then there is no such thing as a ‘universal set’. This is known as **Cantor’s Paradox**.

Cantor's Paradox

Denote $\{x \mid x = x\}$ by U .

U is not a set. (Otherwise $U \notin U$,
or Cantor's Theorem fails to hold for U .)

Russell's Paradox

Denote $\{x \mid x \notin x\}$ by R .

R is not a set. (Otherwise ' $R \in R$ ' is false
and ' $R \notin R$ ' is false.)