1. **Theorem (XI). (Schröder-Bernstein Theorem.)**

   Let A, B be sets. Suppose $A \lesssim B$ and $B \lesssim A$. Then $A \sim B$.

   Remark.

   What is so special about Schröder-Bernstein Theorem?

   - It is usually much more difficult to write down a bijective function than to write down a pair of injective functions. So?

   To verify that two given sets are of cardinality equal to each other, we can choose the easier approach

   > in writing down an injective function from one set to the other, and further writing down an injective function from the latter to the former,

   instead of the more difficult approach in directly writing down a bijective function from one set to the other.

2. **Example (A).**

Another argument for $\mathbb{N} \sim \mathbb{N}^2$.

- Define $f : \mathbb{N} \longrightarrow \mathbb{N}^2$ by $f(x) = (x, 0)$ for any $x \in \mathbb{N}$.

  f is injective. (Exercise.)

  It follows that $\mathbb{N} \lesssim \mathbb{N}^2$.

- Define $g : \mathbb{N}^2 \longrightarrow \mathbb{N}$ by $g(x, y) = 2^x \cdot 3^y$ for any $x, y \in \mathbb{N}$.

  g is injective. (Exercise.)

  It follows that $\mathbb{N}^2 \lesssim \mathbb{N}$.

- Now we have $\mathbb{N} \lesssim \mathbb{N}^2$ and $\mathbb{N}^2 \lesssim \mathbb{N}$.

  According to the Schröder-Bernstein Theorem, $\mathbb{N} \sim \mathbb{N}^2$.

3. **Example (B).**

A simple argument for $\mathbb{N} \sim \mathbb{Q}$.

- We have $\mathbb{N} \subset \mathbb{Q}$. Then $\mathbb{N} \lesssim \mathbb{Q}$.
- We have $\mathbb{Q} \lesssim \mathbb{Z}^2 \sim \mathbb{N}^2 \sim \mathbb{N}$. (Detail for $\mathbb{Q} \lesssim \mathbb{Z}^2$?) Then $\mathbb{Q} \lesssim \mathbb{N}$.
- Now $\mathbb{N} \lesssim \mathbb{Q}$ and $\mathbb{Q} \lesssim \mathbb{N}$.

  According to the Schröder-Bernstein Theorem, $\mathbb{N} \sim \mathbb{Q}$. We also have $\mathbb{N} \sim \mathbb{Z}$ and $\mathbb{Z} \sim \mathbb{Q}$.

Remark.

Hence there are as many natural numbers as there are integers or rational numbers.

4. **Example (C).**

Let S, T be subsets of $\mathbb{R}$.

Suppose S contains as a subset some interval with two or more points.

Suppose T contains as a subset some interval with two or more points.

Then $S \sim T$.

Illustrations through some simple examples.

(C1) $(0,1) \sim [0,1]$.

Justification:

- Define $f : (0,1) \longrightarrow [0,1]$ by $f(x) = x$ for any $x \in (0,1)$.
- Define $g : [0,1] \longrightarrow (0,1)$ by $g(x) = \dfrac{x+1}{3}$ for any $x \in [0,1]$.
- f, g are injective functions. (Exercise.)
  Hence $(0,1) \lesssim [0,1]$ and $[0,1] \lesssim (0,1)$.
  According to the Schröder-Bernstein Theorem, $(0,1) \sim [0,1]$.

Example (C).

Let S, T be subsets of $\mathbb{R}$.

Suppose S contains as a subset some interval with two or more points.

Suppose T contains as a subset some interval with two or more points.

Then $S \sim T$.

Illustrations through some simple examples.

(C1) $(0,1) \sim [0,1]$.

(C2) $[-1,1] \sim \mathbb{R}$.

Justification:

- Define $f : [-1,1] \longrightarrow \mathbb{R}$ by $f(x) = x$ for any $x \in [-1,1]$.
- Define $g : \mathbb{R} \longrightarrow [-1,1]$ by $g(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ for any $x \in \mathbb{R}$.
- f, g are injective. (Exercise.)
  Hence $[-1,1] \lesssim \mathbb{R}$ and $\mathbb{R} \lesssim [-1,1]$.
  According to the Schröder-Bernstein Theorem, $[-1,1] \sim \mathbb{R}$.

Example (C).

Let S, T be subsets of $\mathbb{R}$. Suppose S contains as a subset some interval with two or more points. Suppose T contains as a subset some interval with two or more points. Then $S \sim T$.

Illustrations through some simple examples.

(C1) $(0,1) \sim [0,1]$.

(C2) $[-1,1] \sim \mathbb{R}$.

With a similar argument we can deduce that $I \sim J$ whenever I, J are intervals with at least two points. (Provide the detail.)

Remarks.

- How to prove $[-1,1] \cup (2,3) \sim [-2,0] \cup [1,4)$?
- How about $[1,2] \cup \mathbb{Q} \sim (0.01, 0.09) \cup (0.1, 0.99) \cup \mathbb{N}$?
- How to prove the statement for the general situation?

5. **Example (D).**

Recall that $\mathsf{Map}(\mathbb{N}, [\![0,9]\!])$ is the set of all infinite sequences in $[\![0,9]\!]$.

We argue for $[0,1]\sim\mathsf{Map}(\mathbb{N}, [\![0,9]\!])$:

- For each $r \in [0,1]$, choose one decimal representation of r and write

$$r = 0.r_0r_1r_2r_3\cdots,$$

  and then define the infinite sequence

$$\alpha(r) = (r_0, r_1, r_2, r_3, \cdots).$$

  No two distinct real numbers have the same decimal representation.

  In this way we have defined the injective function

$$\alpha : [0,1] \longrightarrow \mathsf{Map}(\mathbb{N}, [\![0,9]\!]),$$

  given by

$$r \longmapsto \alpha(r) \quad \text{for any } r \in [0,1].$$

  Therefore $[0,1]\lesssim\mathsf{Map}(\mathbb{N}, [\![0,9]\!])$.

Example (D).

Recall that $\mathsf{Map}(\mathbb{N}, [\![0, 9]\!])$ is the set of all infinite sequences in $[\![0, 9]\!]$.

We argue for $[0, 1] \sim \mathsf{Map}(\mathbb{N}, [\![0, 9]\!])$:

- ...

  Therefore $[0, 1] \lesssim \mathsf{Map}(\mathbb{N}, [\![0, 9]\!])$.

- Define the function
$$\rho : \mathsf{Map}(\mathbb{N}, [\![0, 9]\!]) \longrightarrow [0, 1]$$
  by
$$\rho(\{a_n\}_{n=0}^{\infty}) = 0.a_0 5 a_1 5 a_2 5 a_3 5 \cdots \qquad \text{for any } \{a_n\}_{n=0}^{\infty} \in \mathsf{Map}(\mathbb{N}, [\![0, 9]\!]).$$
  ρ is injective. (Exercise.)

  (We can use any one of $1, 2, \cdots, 8$ in place of 5 in the construction of such an injective function.)

  Therefore $\mathsf{Map}(\mathbb{N}, [\![0, 9]\!]) \lesssim [0, 1]$.

- According to the Schröder-Bernstein Theorem, $[0, 1] \sim \mathsf{Map}(\mathbb{N}, [\![0, 9]\!])$.

Example (D).

Recall that $\mathsf{Map}(\mathbb{N}, [\![0,9]\!])$ is the set of all infinite sequences in $[\![0,9]\!]$.

We argue for $[0,1]{\sim}\mathsf{Map}(\mathbb{N}, [\![0,9]\!])$: $\cdots$

Consequences of $[0,1]{\sim}\mathsf{Map}(\mathbb{N}, [\![0,9]\!])$:

(D1) $[0,1]{\sim}\mathsf{Map}(\mathbb{N}, [\![0,9]\!]){\sim}(\mathsf{Map}(\mathbb{N}, [\![0,9]\!]))^2{\sim}[0,1]^2$.

Hence there are as many points in the line segment $[0,1]$ as there are in the square $[0,1]^2$.

(D2) $\mathbb{R}{\sim}[0,1]{\sim}[0,1]^2{\sim}\mathbb{R}^2{\sim}\mathbb{C}$.

There are as many real numbers as there are complex numbers.

(D3) Applying mathematical induction, we have $\mathbb{R}{\sim}\mathbb{R}^n$, $\mathbb{C}{\sim}\mathbb{C}^n$ for any $n \in \mathbb{N}\backslash\{0\}$.

Remarks.

(1) Now it remains to see compare the 'relative sizes' of $\mathbb{Q}$ and $\mathbb{R}$.

(2) What is the significance of $\mathbb{R}{\sim}\mathbb{R}^n$ for any $n \in \mathbb{N}\backslash\{0\}$?

It is that we cannot define 'dimension' by simply comparing the 'relative sizes' of sets.

This surprised Cantor and his contemporaries.

6. **Example (E).**

Let Λ be the set of all lines in $\mathbb{R}^2$. We are going to argue for $\Lambda \sim \mathbb{R}$:

- For each point $(a, b) \in \mathbb{R}^2$, denote by $L_{(a,b)}$ the line given by the equation $y = ax + b$.

  $(a, b) \longmapsto L_{(a,b)}$ defines an injective function from $\mathbb{R}^2$ to Λ.

  Hence $\mathbb{R} \sim \mathbb{R}^2 \lesssim \Lambda$.

- For each line L in $\mathbb{R}^2$, choose one ordered triple (a_L, b_L, c_L) so that L is given by the equation $a_L x + b_L y + c_L = 0$.

  $L \longmapsto (a_L, b_L, c_L)$ defines an injective function from Λ to $\mathbb{R}^3$.

  Hence $\Lambda \lesssim \mathbb{R}^3 \sim \mathbb{R}$.

- Now $\mathbb{R} \lesssim \Lambda$ and $\Lambda \lesssim \mathbb{R}$.

  According to the Schröder-Bernstein Theorem, $\Lambda \sim \mathbb{R}$.

Remark.

With similar arguments, we deduce that the set of all planes in $\mathbb{R}^3$, the set of all circles in $\mathbb{R}^2$, the set of all spheres in $\mathbb{R}^3$ et cetera are of cardinality equal to $\mathbb{R}$.

7. **Preparation for a proof of the Schröder-Bernstein Theorem.**

Recall:

(a) **Definition. (Generalized union and generalized intersection.)**

Let M be a set, and $\{S_n\}_{n=0}^{\infty}$ be an infinite sequence of subsets of the set M.

i. *The* **(generalized) intersection of the infinite sequence of subsets $\{S_n\}_{n=0}^{\infty}$ of the set** *M is defined to be the set $\{x \in M : x \in S_n$ for any $n \in \mathbb{N}\}$. It is denoted by $\bigcap\limits_{n=0}^{\infty} S_n$.*

ii. *The* **(generalized) union of the infinite sequence of subsets $\{S_n\}_{n=0}^{\infty}$ of the set** *M is defined to be the set $\{x \in M : x \in S_n$ for some $n \in \mathbb{N}\}$. It is denoted by $\bigcup\limits_{n=0}^{\infty} S_n$.*

(b) **Theorem (IV). ('Glueing Lemma'.)**

Let A, B be sets. Let $\{C_n\}_{n=0}^{\infty}$, $\{D_n\}_{n=0}^{\infty}$ be infinite sequences of subsets of A, B respectively. Let $\{H_n\}_{n=0}^{\infty}$ be an infinite seuqence of subsets of $A \times B$. Suppose $\{(C_n, D_n, H_n)\}_{n=0}^{\infty}$ is an infinite sequence of bijective functions. Suppose that for any $j, k \in \mathbb{N}$, if $j \neq k$ then $C_j \cap C_k = \emptyset$ and $D_j \cap D_k = \emptyset$. Then $\left(\bigcup\limits_{n=0}^{\infty} C_n, \bigcup\limits_{n=0}^{\infty} D_n, \bigcup\limits_{n=0}^{\infty} H_n\right)$ is a bijective function.

8. **Outline of an argument for the Schröder-Bernstein Theorem.**

Let A, B be sets. Suppose $A \precsim B$ and $B \precsim A$.

Since $A \precsim B$, there is some injective function from A to B, say, $f : A \longrightarrow B$ with graph F.

Since $B \precsim A$, there is some injective function from B to A, say, $g : B \longrightarrow A$ with graph G.

When one of f, g is surjective as well, it will be a bijective function as well. Then we will have $A \sim B$ immediately.

From now on, we assume that neither of f, g is surjective.

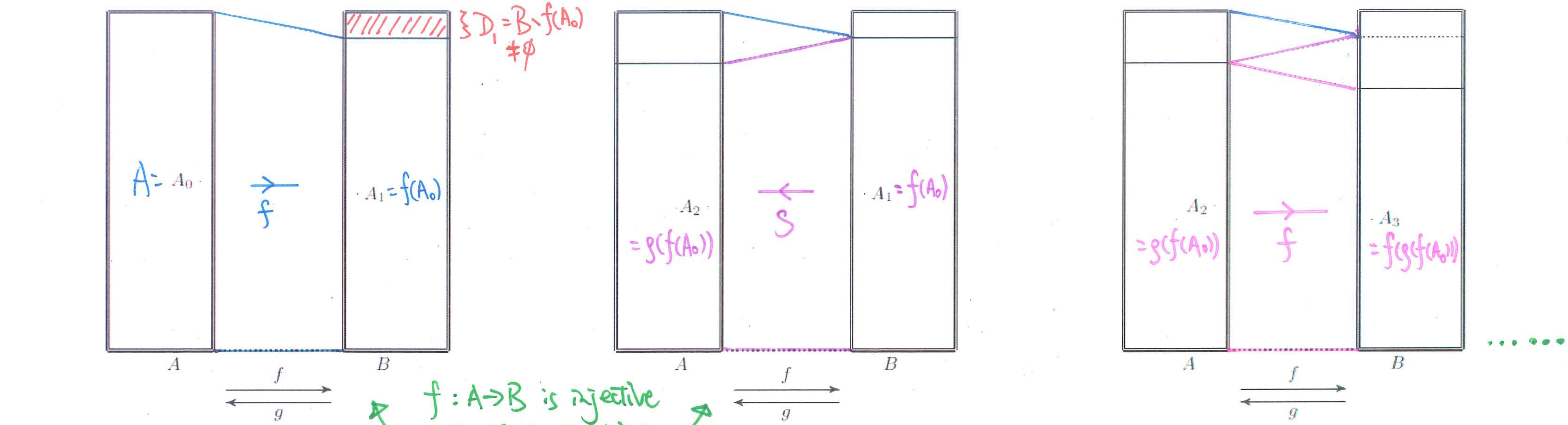
We are going to construct a bijective function from A to B out of f, g.

[*Idea.* Make use of the non-empty sets $B \backslash f(A)$, $A \backslash g(B)$ and the injective functions f, g to 'break up' A, B respectively into many many pieces.

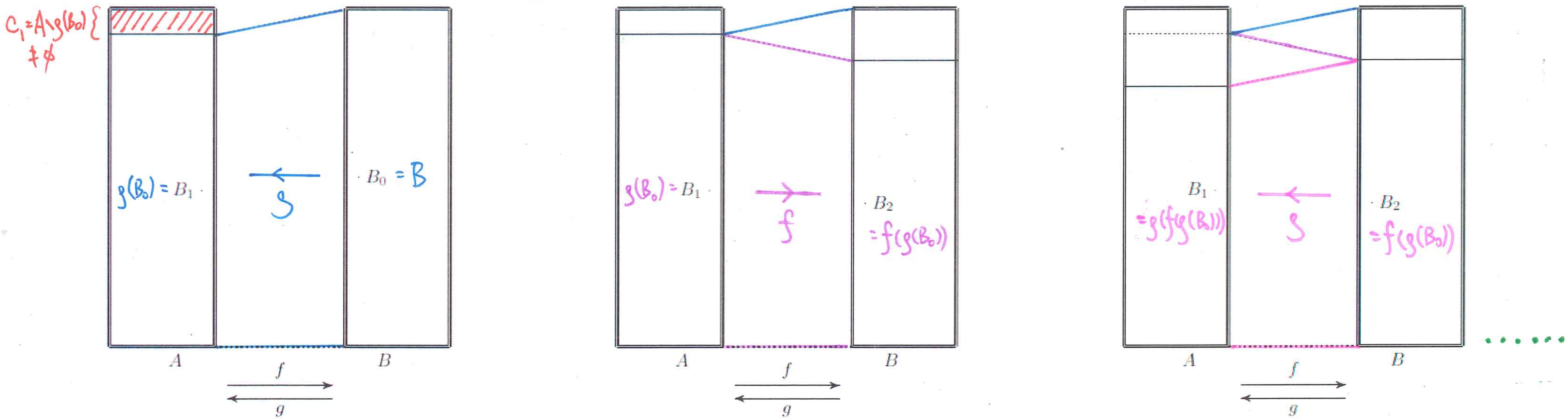
'Arrange' the 'pieces' 'on the two sides' into many many pairs appropriately, with one bijective function defined by f or g as appropriate 'joining' as its domain and range the two 'pieces' in each pair.

'Glue up' the many many bijective functions together to obtain a bijective function from A to B.]

Illustration on the 'non-trivial' situation in the Schröder-Bernstein Theorem.

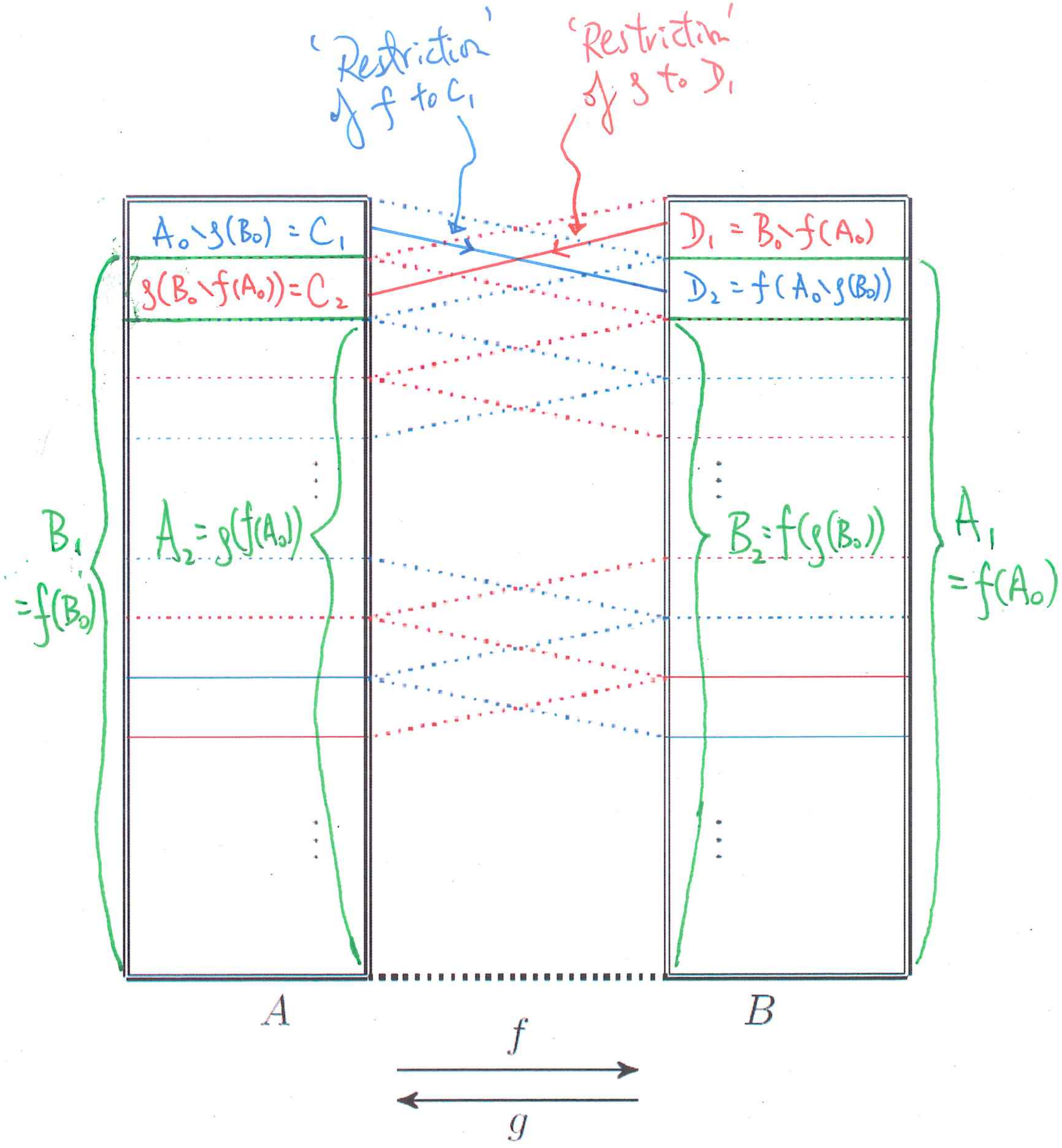


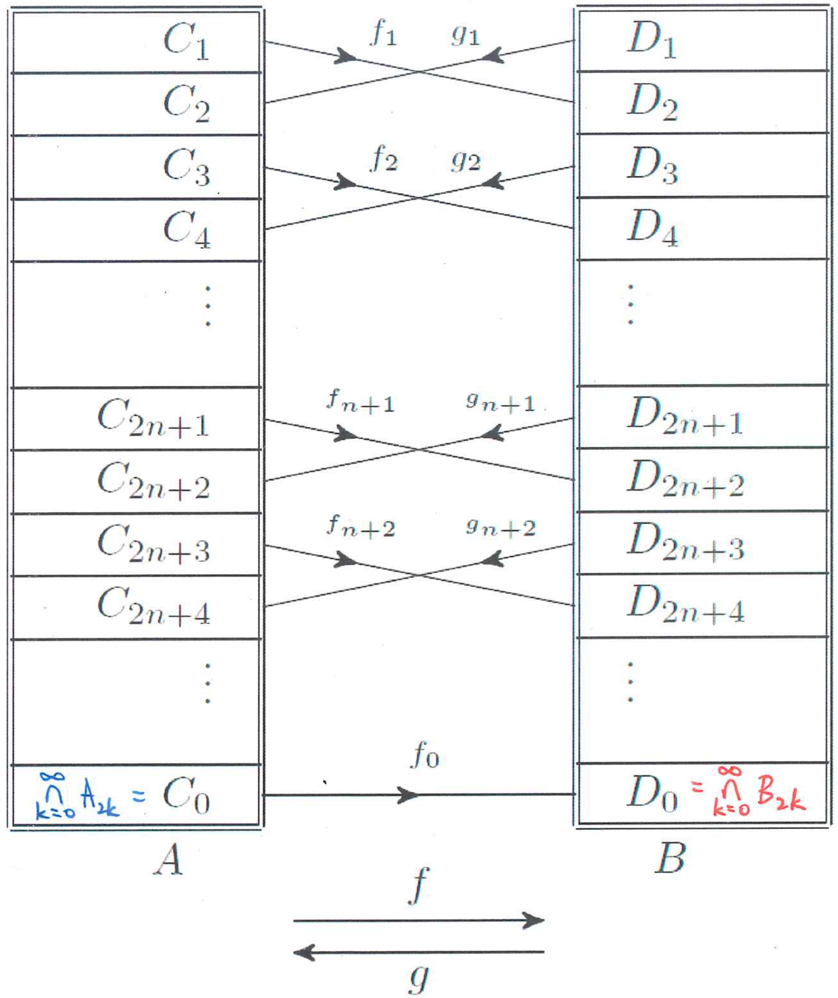
$f: A \to B$ is injective but not surjective. So is $g: B \to A$.



Now juxtapose all diagrams in both infinite sequences of diagrams.

Bijective function from A to B? 'Glue up' the bijective functions $f_1, g_1^{-1}, f_2, g_2^{-1}, \ldots, f_{n+1}, g_{n+1}^{-1}, \ldots$ and f_0.





$C_1 \sim D_2$ 'due to f',

$D_1 \sim C_2$ 'due to g',

so forth and so on.